CURVATURE ON RIEMANNIAN MANIFOLDS AND JACOBI FIELDS

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1. Introduction

Riemannian geometry is a way to study curved spaces by generalizing the familiar Euclidean geometry in $\mathbb{R}^n$ to smooth manifolds, equipped by methods to measure lengths and angles. A natural question is how curvature influences the behavior of geodesics, which are the locally length minimizing paths on geometric objects.

This paper develops an explanation for curvature on Riemannian manifolds. After establishing the necessary background on charts, atlases, and manifolds, the discussion turns to Riemannian metrics, which allow one to measure lengths, angles, and distances on manifolds in a way that generalizes Euclidean geometry. From there we introduce the unique connection determined by the metric and use it as the framework for defining geodesics and relating them to curvature.

With this framework in place, we define curvature via the Riemann curvature tensor and related notions such as sectional curvature, and these concepts are then used to analyze how nearby geodesics deviate from one another. Jacobi fields, which describe variations of geodesics, provide insight into geometric phenomena such as the structure of spaces with negative curvature. The purpose of this paper is to provide an accessible and rigorous overview of how curvature governs the local and global behavior of geodesics on Riemannian manifolds.

2. Manifolds

2.1. **Charts and Atlases (Jonathan).** Throughout this section, we will assume that (M, τ) is a topological space. Thus, M is a set and τ is a collection of open subsets of M satisfying specific properties. See Appendix A for a description of topological spaces.

We would like to study topological spaces (M, τ) that locally resemble $\mathbb{R}^n$. To make this precise, we need to define charts.

Definition 2.1.1 (Charts)**.** *Let (M, τ) be a topological space. A* **n**-dimensional chart *on M is an ordered pair (U, φ) such that*

(1) U is an open subset of M,
(2) $\varphi(U)$ is an open subset of $\mathbb{R}^n$,
(3) $\varphi : U \to \varphi(U)$ is a bijection.

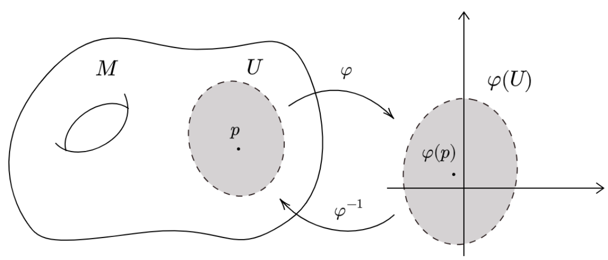

Figure 1. Example chart (U, φ)

Intuitively, for each point p on M, a chart consists of a neighborhood $U \subseteq M$ around p with a map

$$\varphi : U \to \varphi(U) \subset \mathbb{R}^n.$$

For every point p, $\varphi(p) = (x_1, x_2, \ldots, x_n)$ gives the local coordinate representation for p in this chart. A chart demonstrates what it means to locally resemble $\mathbb{R}^n$. Since φ is a bijection between φ and $\varphi(U)$, we can also define a map

$$\varphi^{-1} : \varphi(U) \to U \subset M.$$

Definition 2.1.2 (Compatible Charts)**.** *Let M be a set, and let (U, φ) and (V, ψ) be two n-dimensional charts on M. We call these two charts* compatible *if*

(1) $U \cap V = \emptyset$ or
(2) the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism*, that is, it is bijective and both $\psi \circ \varphi^{-1}$ and its inverse $\varphi \circ \psi^{-1}$ are differentiable.*

If $U \cap V = \emptyset$, then the charts do not overlap, so there are no compatibility issues. If $U \cap V \neq \emptyset$, then the map $\psi \circ \varphi^{-1}$ tells us how to switch between local coordinates on (U, φ) and (V, ψ). For example, if

$$\varphi(p) = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad \psi(p) = (y_1, y_2, \ldots, y_n),$$

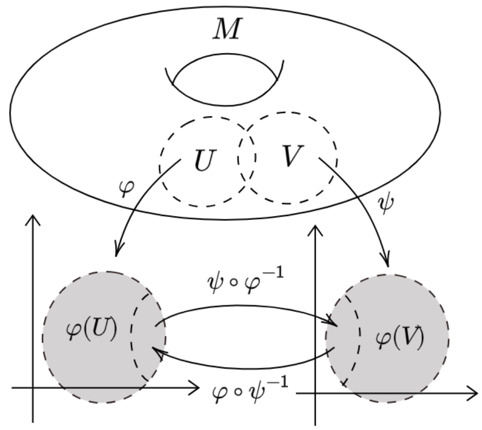

FIGURE 2. Compatible charts (U, φ) and (V, ψ)

where $p \in U \cap V$, then $\psi \circ \varphi^{-1}$ describes how $(x_1, x_2, \ldots, x_n)$ is expressed in terms of $(y_1, y_2, \ldots, y_n)$, and $\varphi \circ \psi^{-1}$ describes how $(y_1, y_2, \ldots, y_n)$ is expressed in terms of $(x_1, x_2, \ldots, x_n)$.

Definition 2.1.3 (Smooth Atlas)**.** *Let M be a set. A* smooth atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ *is a collection of pairwise compatible charts such that*

(1) $\bigcup_{\alpha \in I} U_\alpha = M$*,*
(2) for any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), where $\alpha, \beta \in I$, the map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is smooth (i.e. infinitely differentiable).

An atlas allows us to analyze charts over an entire topological space. In particular, it allows us to use local coordinates everywhere on M.

Definition 2.1.4 (Maximal Atlas)**.** *We call a smooth atlas $\mathcal{A}$ on M* maximal *if it contains every chart on M that is compatible with all charts in $\mathcal{A}$.*

The notion of maximality can be used to describe differentiability for functions $f : M \to \mathbb{R}$. If f is differentiable, then we require $f \circ \varphi^{-1} : U \subseteq \mathbb{R}^n \to \mathbb{R}$ to be differentiable for all charts (U, φ) on a maximal atlas $\mathcal{A}$.

2.2. **Smooth Manifolds (Jonathan).** In this section we define manifolds and smooth manifolds. Informally, a manifold is a space that can be covered by coordinate charts so that each region resembles an open subset of $\mathbb{R}^n$. Requiring charts to be compatible gives a *smooth* manifold, in which differentiation and other tools from calculus can be used in a coordinate-independent way.

Definition 2.2.1 (Manifolds)**.** *An* **n**-dimensional manifold *is a set M along with a maximal atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of n-dimensional charts such that:*

(1) There is a countable collection of charts that cover M.
(2) For any distinct points p and q on M, there exists charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) on M such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$.

Condition (1) tells us that M can be covered by countably many charts. Condition (2), also known as the *Hausdorff property*, tells us that any two distinct points p and q can be separated by two disjoint charts (U, φ) and (V, ψ). We require the Hausdorff property in order to guarantee that convergent sequences have unique limits, a requirement for key theorems on manifolds.

Definition 2.2.2 (Smooth Manifolds)**.** *An n-dimensional manifold M is* smooth *if for any two charts (U, φ) and (V, ψ) on M, the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a C^∞* diffeomorphism*, that is, it is bijective and both $\psi \circ \varphi^{-1}$ and its inverse $\varphi \circ \psi^{-1}$ are* infinitely *differentiable.*

On a smooth manifold M, if two charts (U, φ) and (V, ψ) are compatible and $p \in U \cap V$, then p has local coordinate representations

$$\varphi(p) = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad \psi(p) = (y_1, y_2, \ldots, y_n).$$

related by the change of coordinates given by

$$(y_1, y_2, \ldots, y_n) = (\psi \circ \varphi^{-1})(x_1, x_2, \ldots, x_n)$$

and

$$(x_1, x_2, \ldots, x_n) = (\varphi \circ \psi^{-1})(y_1, y_2, \ldots, y_n).$$

Because these maps are infinitely differentiable, any properties that we describe on the chart (U, φ) holds on the chart (V, ψ). In other words, properties on M are *independent of chart*, which means we can choose any local coordinate system when working on M.

2.3. **Smooth Maps and Smooth Functions (Isabella).** In this subsection we formalize what it means for a function or map on a manifold to be smooth, using the atlas introduced earlier.

Definition 2.3.1 (Smooth function on a manifold)**.** *Let M be a smooth manifold and $k \in \mathbb{N}$. A function $f : M \to \mathbb{R}^k$ is called* smooth *if, for every chart (U, φ) on M, the coordinate representation*

$$f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \to \mathbb{R}^k$$

is a smooth map between open subsets of Euclidean space.

We denote by $C^\infty(M)$ the set of all smooth real-valued functions.

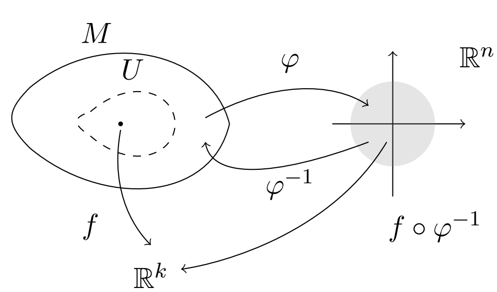

FIGURE 3. Diagram of $f \circ \varphi^{-1}$

Example 2.3.2. *If $M = \mathbb{R}^n$ is smooth, then a function $f : \mathbb{R}^n \to \mathbb{R}$ is smooth in the sense above if and only if it is a smooth function in the usual multivariable-calculus sense. More generally, if $U \subset \mathbb{R}^n$ is open and $M = U$, we recover the standard notion of a smooth map $U \to \mathbb{R}^k$.*

We now extend this notion of smoothness to maps between manifolds.

Definition 2.3.3 (Smooth map between manifolds)**.** *Let M and N be smooth manifolds. A map $F : M \to N$ is called* smooth *if, for every point $p \in M$, there exist charts (U, φ) on M with $p \in U$ and (V, ψ) on N with $F(p) \in V$ and $F(U) \subset V$ such that the coordinate representation*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n$$

is a smooth map between open subsets of Euclidean space.

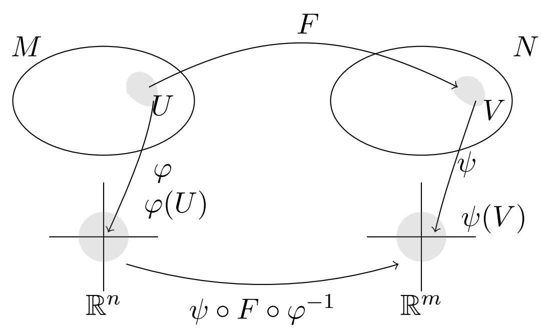

FIGURE 4. Diagram of $\psi \circ F \circ \varphi^{-1}$

Example 2.3.4. *Let S^2 be the unit sphere in $\mathbb{R}^3$. Consider the inclusion map $i : S^2 \to \mathbb{R}^3$ defined by $i(p) = p$ for each point $p \in S^2$. In local coordinates on S^2 (for instance stereographic projection) and standard coordinates on $\mathbb{R}^3$, the map i is given by smooth coordinate functions, so i is a smooth map of manifolds.*

Intuitively, these definitions say that a map between manifolds is smooth exactly when it looks smooth in every compatible coordinate chart. In the next subsection we will use smooth curves $\gamma : (-\varepsilon, \varepsilon) \to M$ and smooth functions $f \in C^\infty(M)$ to define the notion of a tangent vector at a point as a directional derivative.

2.4. **Tangent Space and Vector Fields (Isabella).** Recall from multivariable calculus that given a point $p \in \mathbb{R}^n$ and a direction vector $\mathbf{v} \in \mathbb{R}^n$, the directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction of $\mathbf{v}$ is defined as

$$D_{\mathbf{v}} f(p) = \lim_{h \to 0} \frac{f(p + h\mathbf{v}) - f(p)}{h}.$$

This measures how f changes as we move from p in the direction of $\mathbf{v}$. We will generalize this notion to manifolds.

Definition 2.4.1 (Tangent vector)**.** *Let M be a smooth manifold and $p \in M$. A* tangent vector *at p is a map*

$$v : C^\infty(M) \to \mathbb{R}$$

given by $v(f) = \frac{d}{dt}\big|_{t=0} f(\gamma(t))$ *for some smooth curve* $\gamma : (-\varepsilon, \varepsilon) \to M$ *with* $\gamma(0) = p$*. We write* $v = v_\gamma$ *when we want to emphasize the underlying curve.*

We can think of v_γ as the directional derivative of smooth functions in the direction of the curve γ at p. It satisfies the Leibniz rule

$$v_\gamma(fg) = v_\gamma(f)\, g(p) + f(p)\, v_\gamma(g),$$

so tangent vectors are derivations on $C^\infty(M)$ at the point p.

Definition 2.4.2 (Tangent space)**.** *The* tangent space *at* $p \in M$*, denoted* T_pM*, is the set of all tangent vectors at* p*.*

In local coordinates, T_pM looks exactly like $\mathbb{R}^n$.

Proposition 2.4.3. *Let* $(x_1, \ldots, x_n)$ *be a local coordinate system on* M *near* p*, and let* ∂_i *denote the derivation*

$$\partial_i f = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}\bigg|_{\varphi(p)}.$$

Then $\partial_1, \ldots, \partial_n$ *form a* basis *of the tangent space* T_pM*.*

Example 2.4.4. *For* $M = \mathbb{R}^2$ *with standard coordinates* (x, y)*, the tangent space at any point* p *is naturally identified with* $\mathbb{R}^2$*, and the tangent vectors* ∂_x *and* ∂_y *form a basis of* $T_p\mathbb{R}^2$*.*

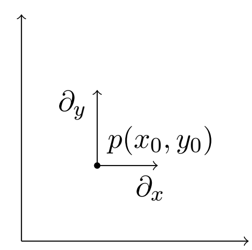

FIGURE 5. ∂_x and ∂_y at p in $\mathbb{R}^2$

Example 2.4.5. *On the sphere* $S^2 \subset \mathbb{R}^3$*, each point* p *has a tangent plane* T_pS^2 *consisting of all vectors in* $\mathbb{R}^3$ *orthogonal to the radius vector at* p*, giving a different* 2*-dimensional tangent space at each point.*

We now smoothly assign a tangent vector—direction that a curve travels in—at every point of the manifold.

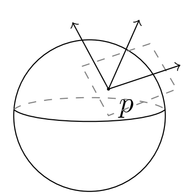

FIGURE 6. Tangent space on S^2

Definition 2.4.6 (Vector field)**.** *A (smooth)* vector field *on M is a map X that assigns to each $p \in M$ a tangent vector $X_p \in T_pM$ such that, for every $f \in C^\infty(M)$, the function*

$$p \to X_p(f)$$

is smooth on M.

Equivalently, in local coordinates $(x_1, \ldots, x_n)$ near p the vector field X can be written as

$$X = a_1(x)\, \partial_{x_1} + \cdots + a_n(x)\, \partial_{x_n},$$

where each coefficient function a_j is smooth. The collection of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

2.5. **Flow generated by a vector field. (Isabella).** An important result in differential geometry is that for any vector field X on a manifold, we can construct a *flow* $\varphi^t : M \to M$ generated by X that satisfies for any $p \in M$ the following properties:

- $\varphi^0(p) = p$ and
- $\partial_t(\varphi^t(p)) = X(\varphi^t(p))$.

To make the existence of such flows φ^t precise, it suffices to understand existence and uniqueness for the corresponding ODE in local coordinates on $\mathbb{R}^n$. This relies on a fixed point argument, so we briefly recall definitions needed to state the two related theorems.

Definition 2.5.1 (Contraction map)**.** *Let (X, d) be a metric space. A map $T : X \to X$ is called a* contraction *(or* contraction mapping*) if there exists a constant $q \in [0, 1)$ such that*

$$d\big(T(x), T(y)\big) \le q\, d(x, y)\,, \forall\, x, y \in X.$$

Definition 2.5.2 (Fixed point iteration, also known as Picard iteration)**.** *Let (X, d) be a metric space and let $T : X \to X$ be a mapping. Given an initial guess $x_0 \in X$, the* fixed point iteration *(also called* Picard iteration*) is the sequence defined by*

$$x_{n+1} = T(x_n), n = 0, 1, 2, \ldots$$

If this sequence converges to some $x^* \in X$ and T is continuous, then x^* satisfies $T(x^*) = x^*$ and is called a fixed point of T.

Example 2.5.3. *Again, on $X = \mathbb{R}$ with the usual metric, the map $T(x) = \frac{1}{2}x$ is a contraction with constant $q = \frac{1}{2}$, since*

$$|T(x) - T(y)| = \tfrac{1}{2}|x - y| \le q\, d(x, y).$$

Starting from any $x_0 \in \mathbb{R}$, the fixed point (Picard) iteration $x_{n+1} = T(x_n)$ produces $x_n = 2^{-n}x_0$, which converges to the unique fixed point $x^* = 0$.

Definition 2.5.4 (Lipschitz continuity)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. The map f is called* Lipschitz continuous *if there exists a constant $K \ge 0$ such that*

$$d_Y\big(f(x), f(y)\big) \le K\, d_X(x, y)\, \forall\, x, y \in X.$$

Any such K is called a Lipschitz constant *for f. If $X, Y \subset \mathbb{R}^n$ with the Euclidean metric, this condition becomes*

$$\|f(x) - f(y)\| \leq K\|x - y\| \, \forall \, x, y \in X.$$

Example 2.5.5. *For Lipschitz continuity, on $X = Y = \mathbb{R}^n$ with the Euclidean norm, the map $f(x) = A(x)$ for a fixed $n \times n$ matrix A is Lipschitz with constant $K = \|A\|$, since*

$$\|f(x) - f(y)\| = \|A(x - y)\| \leq \|A\| \, \|x - y\|.$$

In particular, for $n = 1$ and $f(x) = 3x$, the function is Lipschitz with constant $K = 3$.

Now we present two theorems related to the uniqueness of a flow.

Theorem 2.5.6 (Banach Fixed Point)**.** *Let (X, d) be a non-empty complete metric space and $T : X \to X$ a contraction mapping. Then T has a unique fixed point in X.*

Proof. First, we construct the Picard iteration. Fix any $x_0 \in X$ and define a sequence (x_n) by $x_{n+1} = T(x_n)$, $n = 0, 1, 2, \ldots$ so $x_n = T^n(x_0)$. Next, we estimate successive differences. By the contraction property,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq q \, d(x_n, x_{n-1})$$

for all $n \geq 1$. Iterating this inequality gives

$$d(x_{n+1}, x_n) \leq q^n d(x_1, x_0) \, , \forall \, n \geq 0.$$

We then show (x_n) is Cauchy. For integers $m > n$,

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{m-1} q^k d(x_1, x_0) = d(x_1, x_0) \sum_{k=n}^{m-1} q^k.$$

The geometric sum satisfies

$$\sum_{k=n}^{m-1} q^k \leq \sum_{k=n}^{\infty} q^k = \frac{q^n}{1 - q},$$

so

$$d(x_m, x_n) \leq \frac{q^n}{1 - q} \, d(x_1, x_0).$$

Since $0 \leq q < 1$, the right-hand side tends to 0 as $n \to \infty$, independently of $m > n$. Thus (x_n) is a Cauchy sequence. Because (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*, n \to \infty$. Then, we prove x^* is a fixed point. The map T is continuous (every contraction is Lipschitz with constant $q < 1$), so

$$T(x^*) = T\Big(\lim_{n \to \infty} x_n\Big) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = x^*.$$

Hence x^* is a fixed point of T. Finally, we prove the uniqueness of x^*. Suppose $y^* \in X$ is another fixed point, so $T(y^*) = y^*$. Then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq q \, d(x^*, y^*).$$

Since $0 \leq q < 1$, subtracting $q \, d(x^*, y^*)$ from both sides gives $(1 - q) \, d(x^*, y^*) \leq 0$, so $d(x^*, y^*) = 0$ and hence $x^* = y^*$. Therefore T has a unique fixed point in X. $\square$

Theorem 2.5.7 (Picard–Lindelöf). *Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be open and let $f : D \to \mathbb{R}^n$ be continuous and locally Lipschitz in y, i.e. there is $L > 0$ such that*

$$\|f(t, y_1) - f(t, y_2)\| \le L\|y_1 - y_2\|$$

for all $(t, y_1), (t, y_2) \in D$ with the same first coordinate. Fix $(t_0, y_0) \in D$. Then there exists $\alpha > 0$ and a unique function $y : I_\alpha(t_0) \to \mathbb{R}^n$ solving $y'(t) = f(t, y(t))$, $y(t_0) = y_0$, where $I_\alpha(t_0) = [t_0 - \alpha, t_0 + \alpha]$.

Proof. Choose $a > 0$ and $b > 0$ with $C_{a,b} \subset D$, where

$$C_{a,b} = I_a(t_0) \times B_b(y_0),\ I_a(t_0) = [t_0 - a, t_0 + a],\ B_b(y_0) = \{y : \|y - y_0\| \le b\}.$$

Since $C_{a,b}$ is closed and bounded and f is continuous, the maximum $M = \sup\{\|f(t, y)\| : (t, y) \in C_{a,b}\}$ is finite. Set $\alpha = \min\left\{a, \frac{b}{M}, \frac{1}{2L}\right\}$, and keep $I_\alpha = I_\alpha(t_0)$ fixed from now on.

Consider the Banach space $X = C(I_\alpha, \mathbb{R}^n)$ with the sup norm $\|\varphi\|_\infty = \sup_{t \in I_\alpha} \|\varphi(t)\|$ and the closed ball

$$B_b(y_0) = \{\varphi \in X : \|\varphi(t) - y_0\| \le b\,, \forall t \in I_\alpha\}.$$

Define the Picard operator $\Gamma : X \to X$ by

$$(\Gamma\varphi)(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds,\ t \in I_\alpha.$$

First, Γ maps $B_b(y_0)$ into itself. If $\varphi \in B_b(y_0)$ and $t \in I_\alpha$, then $(s, \varphi(s)) \in C_{a,b}$ for all $s \in I_\alpha$, so $\|f(s, \varphi(s))\| \le M$. Hence

$$\|(\Gamma\varphi)(t) - y_0\| = \left\|\int_{t_0}^{t} f(s, \varphi(s))\, ds\right\| \le \int_{t_0}^{t} \|f(s, \varphi(s))\|\, ds \le \int_{t_0}^{t} M\, ds = M|t - t_0| \le M\alpha \le b,$$

which shows $\Gamma\varphi \in B_b(y_0)$. Next, Γ is a contraction on $B_b(y_0)$. For $\varphi_1, \varphi_2 \in B_b(y_0)$ and $t \in I_\alpha$,

$$\begin{aligned}
&\|(\Gamma\varphi_1)(t) - (\Gamma\varphi_2)(t)\| \\
&= \left\|\int_{t_0}^{t} \big(f(s, \varphi_1(s)) - f(s, \varphi_2(s))\big)\, ds\right\| \\
&\le \int_{t_0}^{t} L\|\varphi_1(s) - \varphi_2(s)\|\, ds \\
&\le L|t - t_0|\|\varphi_1 - \varphi_2\|_\infty.
\end{aligned}$$

Taking the supremum over $t \in I_\alpha$ gives

$$\|\Gamma\varphi_1 - \Gamma\varphi_2\|_\infty \le L\alpha\|\varphi_1 - \varphi_2\|_\infty.$$

By choosing $\alpha \le 1/(2L)$, the constant $q = L\alpha$ satisfies $0 \le q < 1$, so Γ is a contraction on the complete metric space $B_b(y_0) \subset X$. By the Banach fixed point theorem there exists a unique $\varphi \in B_b(y_0)$ with $\Gamma\varphi = \varphi$. For this function,

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds,$$

so φ is differentiable and satisfies $\varphi'(t) = f(t, \varphi(t))$ with $\varphi(t_0) = y_0$ by the fundamental theorem of calculus. Uniqueness of the fixed point of Γ implies uniqueness of the solution in $C(I_\alpha, \mathbb{R}^n)$. This completes the proof. □

3. Riemannian Geometry

3.1. **Riemannian Metric (Isabella).** In this subsection, we present important definitions and results in Riemannian geometry that will be used to prove the theorem at the end of this paper. On a smooth manifold M, we do not have a global ambient notion of length or angle the way we do in $\mathbb{R}^n$. A *Riemannian metric* remedies this by assigning, to each point $p \in M$, an inner product g_p on the tangent space T_pM. Intuitively, g tells us how to measure the sizes of tangent vectors and the angles between them, and it does so *smoothly* as we move from point to point. In this way, a Riemannian metric endows an abstract manifold with geometric structure.

Definition 3.1.1 (Riemannian metric). *A* Riemannian metric $g_p : T_pM \times T_pM \to \mathbb{R}$ *is a bilinear form that satisfies:*

(1) $g_p(u, v) = g_p(v, u)$*;*
(2) $g_p(u, u) \geq 0$ *with equality only when* $u = 0$*;*
(3) g_p *is smooth.*

Property (1) means that the metric is symmetric, and $g(u, u) > 0$ for all nonzero u means that the metric is positive definite. We now introduce a foundational example: the standard metric on $\mathbb{R}^2$.

Example 3.1.2. *Consider the smooth manifold* $\mathbb{R}^2$ *equipped with*

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is called the *standard Euclidean metric* or *dot product* on $\mathbb{R}^2$. At each point $p \in \mathbb{R}^2$, the metric takes two tangent vectors $v = (v_1, v_2)$ and $w = (w_1, w_2)$ and computes $g_p(v, w) = v_1w_1 + v_2w_2$.

Definition 3.1.3 (Riemannian Manifold). *A* Riemannian manifold *is a differentiable manifold* M *equipped with a Riemannian metric* g*.*

Example 3.1.4 (Hyperbolic Plane). *Let* $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ *denote the upper half plane with the metric* $g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$*. Then the hyperbolic plane* $(\mathbb{H}, g)$ *is a Riemannian manifold.*

One of the fundamental applications of a Riemannian metric is that it allows us to define the *length* of a curve on the manifold. Given a smooth curve $\gamma : [a, b] \to M$, its length is given by

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt = \int_a^b \|\dot{\gamma}(t)\|_g \, dt. \tag{3.1}$$

3.2. **Affine Connection (Jonathan).** An affine connection allows us to compare tangent vectors at different points on a manifold. It is the basic tool used later to define geodesics and curvature related quantities.

Definition 3.2.1 (Affine Connection). *Given a smooth manifold* M*, an* affine connection *is an operator* $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ *that satisfies the following properties for any function* $f \in C^\infty(M)$ *and vector fields* $X, Y,$ *and* Z *on* M*:*

(1) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$
(2) $\nabla_{fX+Z}Y = f\nabla_X Y + \nabla_Z Y$
(3) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$

We call ∇ an affine connection or a covariant derivative. Property (1) tells us that ∇ satisfies the Leibniz rule in the upper slot. Property (2) tells us how ∇ is C^∞ linear and additive in the lower slot. Property (3) shows how the upper slot is additive.

Next, we will define a special type of affine connection unique to Riemannian Manifolds.

Definition 3.2.2 (Lie Bracket)**.** *Let M be a smooth manifold and let X and Y be vector fields on M. The* Lie Bracket *of X and Y is the vector field $[X,Y]$ as defined:*

$$[X,Y]f = X(Y(f)) - Y(X(f))$$

for all $f \in C^\infty(M)$.

The Lie Bracket measures how two vector fields vector fields fail to commute. If $X(Y(f)) = Y(X(f))$, then $[X,Y] = 0$. We can also write $[X,Y] = XY - YX$.

Definition 3.2.3 (Levi-Civita connection)**.** *Given a Riemannian Manifold (M,g) and vector fields X,Y,Z on M, the Levi-Civita connection satisfies the following additional properties:*

(1) $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,
(2) $\nabla_X Y - \nabla_Y X = [X,Y]$.

Condition (1) tells us that ∇ is compatible with the Riemannian metric g. In other words, the derivative of the inner product $g(Y,Z)$ in the direction of X can be computed by summing the inner products of g in the directions of Y and Z separately.

Condition (2) tells us how the vector fields X and Y commute on the manifold. By 3.2.2, we can write condition (2) as $\nabla_X Y - \nabla_Y X = [X,Y] = XY - YX$.

3.3. **Christoffel Symbols (Jonathan).** In local coordinates, we can express a connection in terms of functions called Christoffel symbols. Specifically, they account for the changing coordinate basis vectors as we move from point to point.

Definition 3.3.1 (Christoffel Symbols)**.** *Given a connection ∇, local coordinates $x_1, ..., x_n$, and coordinate basis vector fields $\partial_1, \partial_2, \cdots, \partial_n$, the Christoffel Symbols Γ are defined as*

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k.$$

In our Christoffel symbols Γ_{ij}^k, the two lower indices i and j indicate which pair of basis vectors fields ∂_i and ∂_j we are plugging into our connection, while the upper index k tells us which component of the result (along ∂_k) we are measuring.

Lemma 3.3.2 (Symmetry of Christoffel symbols)**.** *Let ∇ be the Levi-Civita connection on (M, g) and let $(x_1, x_2, \dots, x_n)$ be a local coordinate chart with coordinate vector fields $\partial_i = \frac{\partial}{\partial x^i}$. Then, the Christoffel symbols are symmetric on the lower indices:*

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

Proof. By property (2) in 3.2.3, for any vector fields X and Y,

$$\nabla_X Y - \nabla_Y X = [X, Y] = X(Y(f)) - Y(X(f))$$

for all $f \in C^\infty(M)$. For coordinate vector fields

$$X = \frac{\partial}{\partial x_1} = \partial_1, \quad Y = \frac{\partial}{\partial x_2} = \partial_2,$$

$$[X, Y] = [\partial_1, \partial_2] = \frac{\partial^2 f}{\partial x_2 \partial x_1} - \frac{\partial^2 f}{\partial x_1 \partial x_2}.$$

Since $f \in C^\infty(M)$, mixed partials commute, so $[X, Y] = \nabla_X Y - \nabla_Y X = 0$ for all ∂_1, ∂_2. Thus, if $[\partial_1, \partial_2] = 0$, we can write

$$\nabla_{\partial_1} \partial_2 = \nabla_{\partial_2} \partial_1. \tag{3.2}$$

By 3.3.1, we can rewrite both sides as

$$\nabla_{\partial_1} \partial_2 = \sum_{k=1}^n \Gamma_{12}^k \partial_k, \quad \text{and} \quad \nabla_{\partial_2} \partial_1 = \sum_{k=1}^n \Gamma_{21}^k \partial_k,$$

$$\sum_{k=1}^n \Gamma_{12}^k \partial_k = \sum_{k=1}^n \Gamma_{21}^k \partial_k, \tag{3.3}$$

$$\sum_{k=1}^n (\Gamma_{12}^k - \Gamma_{21}^k) \partial_k = 0.$$

Thus $\Gamma_{12}^k = \Gamma_{21}^k$ for all k. □

It is important to note that the Christoffel symbols Γ_{ij}^k are coordinate dependent, so they change with the local coordinates $(x_1, x_2, \dots, x_n)$ of any chart. They are therefore not intrinsic to a manifold M. Rather, they are the coefficients that represent the connection ∇ in any particular chart.

Lemma 3.3.3 (Formula for Christoffel Symbols)**.** *If ∇ is the Levi-Civita connection of a Riemannian manifold (M, g), then the Christoffel symbols are given by*

$$\Gamma_{ij}^k = \sum_{\ell=1}^n \left(\frac{1}{2} g^{k\ell} (\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ji}) \right),$$

where $g^{k\ell}$ is the inverse matrix element of $g_{k\ell}$.

Proof. By property (1) in 3.2.3, we know that for any vector fields $\partial_i, \partial_j, \partial_k$ and metric element g_{ij},

$$\begin{aligned}\partial_k(g_{ji}) &= g(\nabla_{\partial_k}\partial_j, \partial_i) + g(\partial_j, \nabla_{\partial_k}\partial_i)\\ &= \sum_{\ell=1}^{n} g\big(\Gamma^{\ell}_{kj}\partial_\ell, \partial_i\big) + \sum_{\ell=1}^{n} g\big(\partial_j, \Gamma^{\ell}_{ki}\partial_\ell\big)\\ &= \sum_{\ell=1}^{n} \Gamma^{\ell}_{kj}\, g_{\ell i} + \sum_{\ell=1}^{n} \Gamma^{\ell}_{ki}\, g_{j\ell}.\end{aligned} \tag{3.4}$$

By rearranging (i, j, k), the equality above gives

$$\partial_i(g_{kj}) = \sum_{\ell=1}^{n} \Gamma^{\ell}_{ik}\, g_{j\ell} + \sum_{\ell=1}^{n} \Gamma^{\ell}_{ij}\, g_{\ell k} \tag{3.5}$$

$$\partial_j(g_{ki}) = \sum_{\ell=1}^{n} \Gamma^{\ell}_{ji}\, g_{k\ell} + \sum_{\ell=1}^{n} \Gamma^{\ell}_{jk}\, g_{\ell i}. \tag{3.6}$$

By adding equations 3.5 and 3.6 and subtracting 3.4, we can write

$$\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ji} = 2\sum_{r=1}^{n} \Gamma^{r}_{ij}\, g_{rk} \tag{3.7}$$

where we noted that by 3.3.2 Christoffel symbols are symmetric on the lower indices.

Switching our free variable from k to ℓ, we obtain

$$2\sum_{r=1}^{n} \Gamma^{r}_{ij}\, g_{r\ell} = \partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ji}.$$

Fix an index $m \in \{1, 2 \ldots, n\}$. Multiplying both sides by $\frac{1}{2}g^{m\ell}$ and summing over ℓ:

$$\sum_{\ell=1}^{n} \frac{1}{2} g^{m\ell} \cdot 2\sum_{r=1}^{n} \Gamma^{r}_{ij}\, g_{r\ell} = \sum_{\ell=1}^{n} \left(\frac{1}{2} g^{m\ell}(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ji})\right)$$

Looking at the LHS of the equation, g^{ml} is the inverse matrix to g_{ml}, so

$$\sum_{\ell=1}^{n} g^{m\ell} g_{r\ell} = \delta^m_r.$$

Hence the LHS becomes

$$\sum_{k=1}^{n} \Gamma^{k}_{ij}\delta^m_k = \Gamma^m_{ij}.$$

Switching our dummy variable from m to k, we get

$$\Gamma^k_{ij} = \sum_{\ell=1}^{n} \left(\frac{1}{2} g^{k\ell}(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ji})\right).$$

□

This shows how the Christoffel Symbols for any Levi-Civita connection are determined by metric elements g_{ij}. This leads us to the following lemma:

Lemma 3.3.4 (Uniqueness of Levi-Civita connection)**.** *Every Riemannian manifold has a unique Levi-Civita connection.*

Proof. In 3.3.3, we have shown how the Christoffel symbols for any Levi-Civita connection ∇ are completely determined by our metric elements g_{ij}. Thus, if we have any two Levi-Civita connections ∇_1 and ∇_2, then their Christoffel symbols must be equivalent on every chart. Thus the Levi-Civita connection is unique. ☐

Additionally, there exists an alternative proof without using 3.3.3. It comes from an alternative definition of the Levi-Civita connection entirely.

Proof. Given a Riemannian Manifold (M, g) and vector fields X, Y, and Z on M, by metric compatibility, we can write

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{3.8}$$

$$Yg(Z,X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \tag{3.9}$$

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{3.10}$$

We can thus compute

$$\begin{aligned} Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) &= g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y) + g(\nabla_X Y + \nabla_Y X, Z) \\ &= g(Y, [X,Z]) + g(X, [Y,Z]) + g(Z, [Y,X]) + 2\, g(Z, \nabla_X Y). \end{aligned}$$

$$\implies \quad 2\, g(Z, \nabla_X Y) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(Y,[X,Z]) - g([Y,Z],x) + g(Z,[Y,X]).$$

where ∇ is the Levi-Civita connection and $[\cdot,\cdot]$ denotes the Lie Bracket. This equation implies that the Levi-Civita connection is unique because it determines $g(Z, \nabla_X Y)$ for all X, Y, Z. ☐

We now illustrate a computation of Christoffel symbols using a concrete example:

Example 3.3.5. *We compute Γ^k_{ij} for the hyperbolic space defined by $\mathbb{H}$ and metric g where*

$$\mathbb{H} = \{(x,y) \in \mathbb{R} : y > 0\}, \quad g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

We use coordinates $(x_1, x_2) = (x, y)$ and

$$\frac{\partial}{\partial x_1} = \partial_x = \partial_1, \quad \frac{\partial}{\partial x_2} = \partial_y = \partial_2.$$

We start by calculating

$$g^{-1} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix},$$

thus $g^{11} = g^{22} = y^2$ and $g^{12} = g^{21} = 0$. Using 3.3.3, we obtain

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y}, \quad \Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^2_{22} = -\frac{1}{y}$$

with the rest $\Gamma^1_{11}, \Gamma^1_{22}, \Gamma^2_{12}, \Gamma^2_{21}$ *equal to* 0.

3.4. **Geodesics (Isabella).** Geodesics are curves that generalize straight lines in the standard Euclidean plane to Riemannian manifolds. The Levi–Civita connection allows us to define geodesics. From here on let ∇ denote the Levi-Civita connection associated to the Riemannian metric g on M.

Definition 3.4.1 (Geodesic)**.** *A smooth curve* $\gamma : (a, b) \to M$ *is called a* geodesic *if its covariant acceleration vanishes, in other words* $\nabla_{\dot\gamma}\dot\gamma = 0$.

Lemma 3.4.2. *If* γ *is a geodesic, then its speed* $\langle \dot\gamma(t), \dot\gamma(t)\rangle_{g_{\gamma(t)}}$ *is constant.*

Proof. We can see this directly using metric compatibility of the Levi–Civita connection:

$$\frac{d}{dt}\big\langle \dot\gamma(t), \dot\gamma(t)\big\rangle_g = (\nabla_{\dot\gamma} g)(\dot\gamma, \dot\gamma) + 2\big\langle \nabla_{\dot\gamma}\dot\gamma, \dot\gamma\big\rangle_g.$$

For the Levi–Civita connection we have $\nabla g = 0$, and for a geodesic $\nabla_{\dot\gamma}\dot\gamma = 0$, so the right-hand side vanishes and $\langle \dot\gamma(t), \dot\gamma(t)\rangle_g$ is constant in t. □

Recalling that the length of a smooth curve $\gamma : [a, b] \to M$ is given by Equation (3.1), we often reparametrize geodesics so that they have unit speed, in which case γ is parametrized by arc length.

Proposition 3.4.3 (Local minimizing property of geodesics)**.** *Let* (M, g) *be a Riemannian manifold. Then for every point* $p \in M$ *there exists a neighborhood* U *of* p *such that if* $q \in U$*, the unique geodesic segment from* p *to* q *realizes the minimal length among all smooth curves joining* p *and* q.

In particular, if $p, q \in M$ are sufficiently close, the geodesic segment between p and q is the shortest path with respect to the metric g.

3.5. **Geodesics on the hyperbolic plane.** Consider the upper half-plane model of the hyperbolic plane,

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\},$$

equipped with the metric

$$g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

We compute the equations of geodesics in the Hyperbolic plane. The nonzero Christoffel symbols (with $1 = x$, $2 = y$) are

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y},\ \Gamma^2_{11} = \frac{1}{y},\ \Gamma^2_{22} = -\frac{1}{y}.$$

The geodesic equation in local coordinates is

$$\ddot x^\ell + \Gamma^\ell_{jk}\,\dot x^j \dot x^k = 0,\ \ell = 1, 2,$$

where $\gamma(t) = (x(t), y(t))$ and $\dot x^1 = \dot x$, $\dot x^2 = \dot y$.

For $\ell = 1$, only the terms with Γ^1_{12} and Γ^1_{21} contribute, so we have

$$\ddot x + \Gamma^1_{11}\dot x^2 + 2\Gamma^1_{12}\dot x\dot y + \Gamma^1_{22}\dot y^2 = \ddot x + 2\Gamma^1_{12}\dot x\dot y = \ddot x - \frac{2}{y}\dot x\dot y = 0.$$

For $\ell = 2$, we have Γ^2_{11} and Γ^2_{22}:

$$\ddot{y} + \Gamma^2_{11}\dot{x}^2 + 2\Gamma^2_{12}\dot{x}\dot{y} + \Gamma^2_{22}\dot{y}^2 = \ddot{y} + \Gamma^2_{11}\dot{x}^2 + \Gamma^2_{22}\dot{y}^2 = \ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 = 0.$$

So a curve $\gamma(t) = (x(t), y(t))$ in $\mathbb{H}^2$ is a geodesic if and only if it satisfies the geodesic equation

$$\begin{cases} \ddot{x} - \dfrac{2}{y}\dot{x}\dot{y} = 0, \\ \ddot{y} + \dfrac{1}{y}\dot{x}^2 - \dfrac{1}{y}\dot{y}^2 = 0. \end{cases}$$

The solutions of this system that lie in $\mathbb{H}^2$ are precisely vertical rays with an open end on the x–axis and semicircles contained in $\mathbb{H}^2$ that are perpendicular to the x-axis. Each such curve can be parametrized with constant hyperbolic speed, so these are exactly the geodesics of the hyperbolic plane in the upper half-plane model.

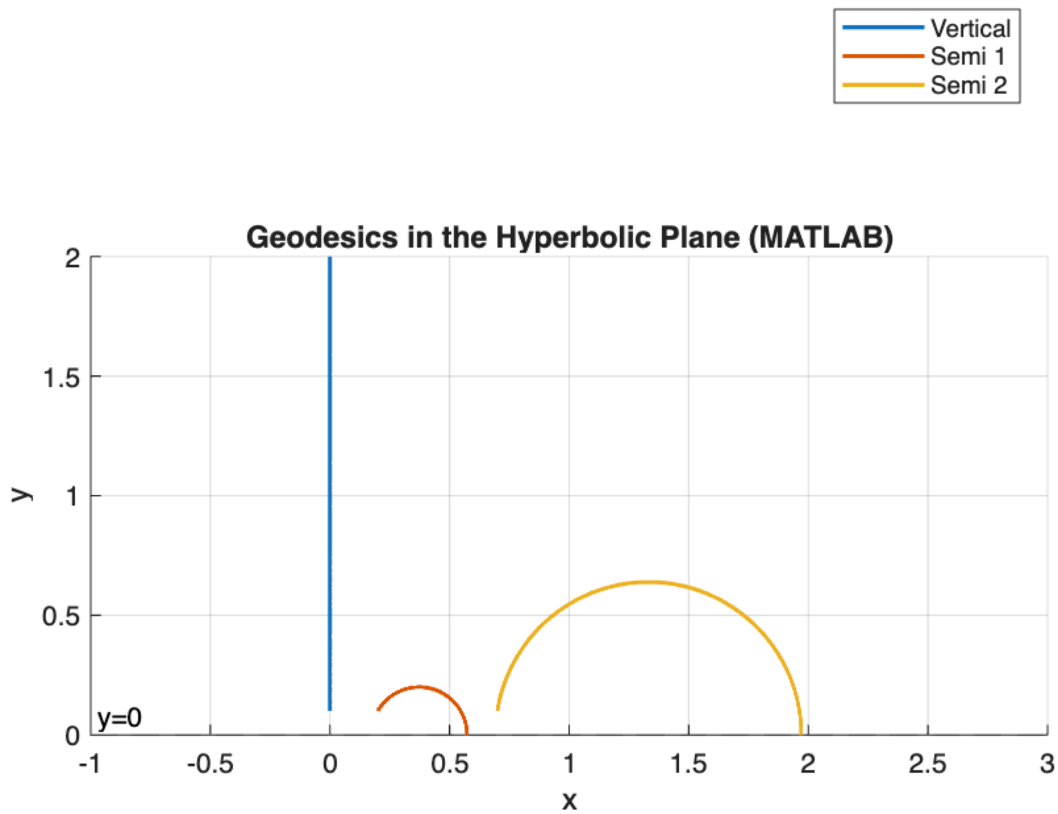

FIGURE 7. Vertical and semicircle geodesics

4. CURVATURE

4.1. **Riemann Curvature Tensor (Milind).** Having discussed smooth manifolds as well as Riemannian metrics, which prescribe a way to measure distances and angles on a smooth manifold, we are properly equipped to quantify a smooth manifold's curvature. The single object that contains all information about curvature is called the Riemann curvature tensor.

Definition 4.1.1. (Riemann Curvature Tensor). *Let M be a smooth manifold with an affine connection ∇. The Riemann curvature tensor $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined for $X, Y, Z \in \mathfrak{X}(M)$ by*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Remark. *Sometimes we will refer to $R(u,v)w$ where $u, v, w \in T_pM$ are vectors for some $p \in M$ (they are not vector fields). This can be interpreted as follows: construct $U, V, W \in \mathfrak{X}(M)$ with $U_p = u, V_p = v, W_p = w$. Then $R(u,v)w$ is the vector field $R(U,V)W$ evaluated at p.*

We now briefly present a property of the Riemann curvature tensor that illustrate its usefulness. In fact, it is the sole reason it earns the name "tensor".

Proposition 4.1.2. *Let M be a smooth manifold with a connection ∇ and Riemann curvature tensor R. Then $R(X,Y)Z$ is $C^\infty(M)$-linear in each argument.*

Proof. First, we prove $R(Y,X)Z = -R(X,Y)Z$. Note

$$R(Y,X)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z = -(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) - \nabla_{-[X,Y]} Z,$$

and since $\nabla_{-[X,Y]} Z = -\nabla_{[X,Y]} Z$, the above rearranges to

$$-(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) = -R(X,Y)Z.$$

Now, consider the proposition.

Having proven $R(X,Y)Z = -R(Y,X)Z$, to show that R is $C^\infty(M)$-linear in all three slots, it suffices to show linearity in the first and third slots. For the first slot, it suffices to show $R(V + fW, Y)Z = R(V,Y)Z + fR(W,Y)Z$ for any $V, W \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. This follows from a computation:

$$\begin{aligned} R(V + fW, Y)Z &= \nabla_{V+fW} \nabla_Y Z - \nabla_Y \nabla_{V+fW} Z - \nabla_{[V+fW,Y]} Z \\ &= (\nabla_V \nabla_Y Z + f\nabla_W \nabla_Y Z) - \nabla_Y (\nabla_V Z + f\nabla_W Z) - \nabla_{[V,Y]+[fW,Y]} Z \\ &= R(V,Y)Z + f\nabla_W \nabla_Y Z - f\nabla_Y \nabla_W Z - Y(f)\nabla_W Z - \nabla_{f[W,Y]-Y(f)W} Z \\ &= R(V,Y)Z + f\nabla_W \nabla_Y Z - f\nabla_Y \nabla_W Z - f\nabla_{[W,Y]} Z - Y(f)\nabla_W Z + Y(f)\nabla_W Z \\ &= R(V,Y)Z + fR(W,Y)Z. \end{aligned}$$

For linearity of the third slot, $R(X,Y)(W+V) = R(X,Y)V + R(X,Y)W$ is clear from the linearity of the connection, so it remains to show $R(X,Y)(fZ) = fR(X,Y)Z$ for $f \in C^\infty(M)$. Indeed,

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_X \nabla_Y fZ - \nabla_Y \nabla_X fZ - \nabla_{[X,Y]} fZ \\ &= X(Y(f))Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z + f\nabla_X \nabla_Y Z \\ &- Y(X(f))Z - X(f)\nabla_Y Z - Y(f)\nabla_X Z - f\nabla_Y \nabla_X Z \\ &- X(Y(f))Z + Y(X(f))Z - f\nabla_{[X,Y]} Z \\ &= f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - f\nabla_{[X,Y]} Z = fR(X,Y)Z, \end{aligned}$$

as desired. ∎

An important consequence of Proposition 4.1.2 is that to compute the Riemann curvature tensor R on any three vector fields, it suffices to compute $R(\partial_i, \partial_j)\partial_k$ for all positive integers $i, j, k \leq \dim(M)$, and we take linear combinations of these values.

Another important note regarding practical applications of the Riemann curvature tensor applies when (M, g) is a Riemannian manifold. Then, when we say "the" Riemann curvature tensor, we refer to the Riemann curvature tensor in Definition 4.1.1 using the Levi-Civita connection as ∇. This is because, as discussed previously, the Levi-Civita connection is the most natural connection on a Riemann manifold in many ways.

We now present a few properties of the Riemann curvature tensor that help us algebraically manipulate it as well as compute its values.

Proposition 4.1.3. *Let (M, g) be a Riemannian manifold with Riemann curvature tensor R. Then if $p \in M$, then for any $x, y, u, v \in T_pM$ we have:*

(1) $R(x, y)u = -R(y, x)u$,
(2) $\langle R(x, y)u, v\rangle_g = -\langle R(x, y)v, u\rangle_g$,
(3) $\langle R(x, y)u, v\rangle_g = \langle R(y, x)v, u\rangle_g$.

Proof. Due to Proposition 4.1.2, it suffices to prove each of the statements only when x, y, u, v are basis vectors (the general statements then follow by linearity). For the first statement, let $\partial_i, \partial_j, \partial_k$ be basis vectors and observe that

$$R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{[\partial_i,\partial_j]}\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k,$$

from which it is clear that swapping i and j negates the expression. Thus, $R(\partial_i, \partial_j)\partial_k = -R(\partial_j, \partial_i)\partial_k$, proving statement 1. Observe that statement 3 follows from statements 1 and 2, so it suffices to prove statement 2.

First, we prove the statement $\langle R(X, Y)Z, Z\rangle_g = 0$ for vector fields X, Y, Z. Recall that metric compatibility condition states that for vector fields V_1, V_2, V_3

$$V_1\langle V_2, V_3\rangle_g = \langle \nabla_{V_1}V_2, V_3\rangle_g + \langle V_2, \nabla_{V_1}V_3\rangle_g.$$

Consider the above equation with the three triples of vector fields $(V_1, V_2, V_3) = (X, \nabla_Y Z, Z)$, $(Y, \nabla_X Z, Z)$, and $([X, Y], Z, Z)$. We obtain the following three equations.

$$X\langle \nabla_Y Z, Z\rangle_g = \langle \nabla_X\nabla_Y Z, Z\rangle_g + \langle \nabla_Y Z, \nabla_X Z\rangle_g$$
$$Y\langle \nabla_X Z, Z\rangle_g = \langle \nabla_Y\nabla_X Z, Z\rangle_g + \langle \nabla_X Z, \nabla_Y Z\rangle_g$$
$$[X, Y]\langle Z, Z\rangle_g = 2\langle \nabla_{[X,Y]}Z, Z\rangle_g.$$

Subtracting the second equation from the first, and then subtracting half of the third equation yields:

$$X\langle \nabla_Y Z, Z\rangle_g - Y\langle \nabla_X Z, Z\rangle_g - \frac{1}{2}[X, Y]\langle Z, Z\rangle_g = \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, Z\rangle_g,$$
$$\implies X\langle \nabla_Y Z, Z\rangle_g - Y\langle \nabla_X Z, Z\rangle_g - \frac{1}{2}[X, Y]\langle Z, Z\rangle_g = \langle R(X, Y)Z, Z\rangle_g.$$

Using metric compatibility with $(V_1, V_2, V_3) = (X, Z, Z)$ now gives $X\langle Z, Z\rangle_g = 2\langle\nabla_X Z, Z\rangle_g$, so $\langle\nabla_X Z, Z\rangle_g = X\langle Z, Z\rangle_g/2$. Likewise, $\langle\nabla_Y Z, Z\rangle_g = Y\langle Z, Z\rangle_g/2$. Substituting into the above display yields

$$\frac{XY}{2}\langle Z, Z\rangle_g - \frac{YX}{2}\langle Z, Z\rangle_g - \frac{[X, Y]}{2}\langle Z, Z\rangle_g = \langle R(X, Y)Z, Z\rangle_g.$$

But the left hand side is simply the operator $\frac{XY}{2} - \frac{YX}{2} - \frac{[X,Y]}{2} = \frac{XY}{2} - \frac{YX}{2} - \frac{XY - YX}{2} = 0$ applied to $\langle Z, Z\rangle_g$. This is 0. Hence, the right hand side $\langle R(X, Y)Z, Z\rangle_g$ is 0.

To finally prove statement 2 of the proposition, consider the following for vector fields X, Y, Z, W:

$$\begin{aligned}
0 &= \langle R(X, Y)(Z + W), Z + W\rangle_g \\
&= \langle R(X, Y)Z, Z\rangle_g + \langle R(X, Y)W, Z\rangle_g + \langle R(X, Y)Z, W\rangle_g + \langle R(X, Y)W, W\rangle_g \\
&= \langle R(X, Y)W, Z\rangle_g + \langle R(X, Y)Z, W\rangle_g,
\end{aligned}$$

where the second line follows from linearity of R. We now have $\langle R(X, Y)W, Z\rangle_g + \langle R(X, Y)Z, W\rangle_g = 0$, or equivalently $\langle R(X, Y)W, Z\rangle_g = -\langle R(X, Y)Z, W\rangle_g$ as desired.

□

We now provide a couple of examples of the Riemann curvature tensor.

Example 4.1.4. *Consider the smooth manifold $\mathbb{R}^2$ equipped with its standard metric $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We can compute from Lemma 3.3.3 that*

$$\Gamma_{ij}^k = \sum_{\ell=1}^{2} \frac{g^{k\ell}}{2}(\partial_i g_{\ell j} + \partial_j g_{\ell i} - \partial_\ell g_{ij}) = 0$$

for all $i, j, k \in \{1, 2\}$ since each metric component $g_{11}, g_{12}, g_{21}, g_{22}$ is constant and thus has zero derivative. Since all of the Christoffel symbols are 0, we have $\nabla_{\partial_j}\partial_i = 0$ for each pair $i, j \in \{1, 2\}$.

We now compute $R(\partial_i, \partial_j)\partial_k$ for any positive integers $i, j, k \in \{1, 2\}$. Since $[\partial_i, \partial_j] = 0$ here, we have

$$R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k = \nabla_{\partial_i}0 - \nabla_{\partial_j}0 = 0.$$

Hence, $R(X, Y, Z)$, being a linear combination of the values $R(\partial_i, \partial_j)\partial_k$, is always 0. In other words, $R = 0$ for this specific Riemannian manifold. As $\mathbb{R}^2$ under its standard metric is known to be flat (imagine it as, say, an infinite flat piece of paper), the curvature tensor being 0 here is promising as to why this is a reasonable definition of curvature.

Example 4.1.5. *Consider the hyperbolic space $\mathbb{H}^2 = \{(x, y)|y > 0\}$ with $g = \begin{pmatrix} y^{-2} & 0 \\ 0 & y^{-2} \end{pmatrix}$. From Example 3.3.5 we have the Christoffel symbols on this manifold. Now, from Definition 3.3.1, we can write*

$$\nabla_{\partial_1}\partial_1 = \frac{\partial_2}{y}, \nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = -\frac{\partial_1}{y}, \nabla_{\partial_2}\partial_2 = -\frac{\partial_2}{y}.$$

Noting that the basis vector fields commute here, we have that the Lie bracket of basis vector fields is 0. Thus, $R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k$. Using the results in the above display, we can then *carry out the derivatives to compute* $R(\partial_i, \partial_j)\partial_k$ *for every three* $i, j, k \in \{1, 2\}$*. This yields*

$$R(\partial_1, \partial_2)\partial_1 = \frac{\partial_2}{y^2}, R(\partial_2, \partial_1)\partial_1 = -\frac{\partial_2}{y^2},$$

$$R(\partial_1, \partial_2)\partial_2 = -\frac{\partial_1}{y^2}, R(\partial_2, \partial_1)\partial_2 = \frac{\partial_1}{y^2},$$

with the rest being 0*. As we can see, the Riemann curvature tensor is nonzero on this manifold. Yet, the curvature is still quite simple. In general, it could be even more complicated.*

4.2. **Sectional Curvature (Milind).** The Riemann curvature tensor provides a complete picture of a Riemannian manifold's curvature. However, it can be unwieldy as it takes three vector inputs, and furthermore it outputs a vector rather than a scalar. Thus, we introduce a notion of curvature that is simpler. It will be a scalar that only depends on two vector inputs.

Definition 4.2.1. (Sectional Curvature). *Let* (M, g) *be a Riemannian manifold with Riemann curvature tensor* R*. If* $p \in M$ *and* $u, v \in T_pM$ *are linearly independent, then the sectional curvature* $K(u, v)$ *is defined by*

$$K(u, v) = \frac{\langle R(u, v)v, u\rangle_g}{\langle u, u\rangle_g \langle v, v\rangle_g - \langle u, v\rangle_g^2}$$

Note that by creating this simpler definition of curvature, a price is paid of losing some information about the original Riemann curvature tensor. In particular, while we can always compute the sectional curvature from the Riemann curvature tensor, the converse is clearly not true. That being said, the sectional curvature does have some convenient properties.

Lemma 4.2.2. *Let* (M, g) *be a Riemannian manifold and let* K *denote the sectional curvature. Suppose* $p \in M$ *and* $\mathcal{P}$ *is a two dimensional subspace of* T_pM*. Then* $K(u, v)$ *is the same for any linearly independent* $u, v \in \mathcal{P}$*.*

Proof. Let e_1, e_2 be an orthonormal basis of $\mathcal{P}$. In other words, $\langle e_1, e_1\rangle_g = \langle e_2, e_2\rangle_g = 1$ and $\langle e_1, e_2\rangle_g = 0$. It suffices to show that $K(u, v) = K(e_1, e_2)$ for any linearly independent $u, v \in \mathcal{P}$. We can in fact compute

$$K(e_1, e_2) = \frac{\langle R(e_1, e_2)e_2, e_1\rangle_g}{\langle e_1, e_1\rangle_g \langle e_2, e_2\rangle_g - \langle e_1, e_2\rangle_g^2} = \frac{\langle R(e_1, e_2)e_2, e_1\rangle_g}{1 \cdot 1 - 0^2} = \langle R(e_1, e_2)e_2, e_1\rangle_g.$$

Thus, it suffices to show $K(u, v) = \langle R(e_1, e_2)e_2, e_1\rangle_g$.

Since $\mathcal{P}$ is the span of e_1, e_2, we can write

$$u = Ae_1 + Be_2$$

$$v = Ce_1 + De_2$$

for some scalars A, B, C, D. Note that the condition the u, v are linearly independent is equivalent to $A/C \neq B/D$, or equivalently $AD - BC \neq 0$.

Using linearity of the metric we can compute the following:

$$\langle u, u\rangle_g = \langle Ae_1 + Be_2, Ae_1 + Be_2\rangle_g = A^2\langle e_1, e_1\rangle_g + B^2\langle e_2, e_2\rangle_g + 2AB\langle e_1, e_2\rangle_g = A^2 + B^2$$

$$\langle v,v\rangle_g = \langle Ce_1+De_2, Ce_1+De_2\rangle_g = C^2\langle e_1,e_1\rangle_g + D^2\langle e_2,e_2\rangle_g + 2CD\langle e_1,e_2\rangle_g = C^2+D^2$$

$$\langle u,v\rangle_g = \langle Ae_1+Be_2, Ce_1+De_2\rangle_g = AC\langle e_1,e_1\rangle_g + BD\langle e_2,e_2\rangle_g + (AD+BC)\langle e_1,e_2\rangle_g = AC+BD.$$

Hence,

$$\langle u,u\rangle_g\langle v,v\rangle_g - \langle u,v\rangle_g^2 = (A^2+B^2)(C^2+D^2)-(AC+BD)^2 = A^2D^2+B^2C^2-2ABCD = (AD-BC)^2.$$

We can also write

$$\langle R(u,v)v,u\rangle_g = \langle R(Ae_1+Be_2, Ce_1+De_2)(Ce_1+De_2), Ae_1+Be_2\rangle_g.$$

Recall that each slot of the Riemann tensor is linear and each slot of the metric is linear. Thus we can write the above as a linear combination of the 16 values $\langle R(e_i,e_j)e_k,e_\ell\rangle_g$ for $i,j,k,\ell \in \{1,2\}$. But note that if $e_j = e_i$ then from Proposition 4.1.3, $R(e_i,e_i)e_k = -R(e_i,e_i)e_k$, so $R(e_i,e_i)e_k = 0$, meaning $\langle R(e_i,e_j)e_k,e_\ell\rangle_g = 0$. Similarly, if $e_\ell = e_k$ then $\langle R(e_i,e_j)e_k,e_k\rangle_g = -\langle R(e_i,e_j)e_k,e_k\rangle_g$, so $\langle R(e_i,e_j)e_k,e_k\rangle_g = 0$ again. Hence, only 4 nonzero terms remain:

$$\begin{aligned}\langle R(u,v)v,u\rangle_g &= ADCB\langle R(e_1,e_2)e_1,e_2\rangle_g + BCCB\langle R(e_2,e_1)e_1,e_2\rangle_g \\ &\quad + ADDA\langle R(e_1,e_2)e_2,e_1\rangle_g + BCDA\langle R(e_2,e_1)e_2,e_1\rangle_g.\end{aligned}$$

Note that $\langle R(e_1,e_2)e_1,e_2\rangle_g = \langle R(e_2,e_1)e_2,e_1\rangle_g = -\langle R(e_1,e_2)e_2,e_1\rangle_g$ and $\langle R(e_2,e_1)e_1,e_2\rangle_g = \langle R(e_1,e_2)e_2,e_1\rangle_g$, by Proposition 4.1.3. Thus,

$$\begin{aligned}\langle R(u,v)v,u\rangle_g &= (-ADCB+BCCB+ADDA-BCDA)\langle R(e_1,e_2)e_2,e_1\rangle_g \\ \implies \langle R(u,v)v,u\rangle_g &= (AD-BC)^2\langle R(e_1,e_2)e_2,e_1\rangle_g.\end{aligned}$$

We can now compute

$$K(u,v) = \frac{\langle R(u,v)v,u\rangle_g}{\langle u,u\rangle_g\langle v,v\rangle_g - \langle u,v\rangle_g^2} = \frac{(AD-BC)^2\langle R(e_1,e_2)e_2,e_1\rangle_g}{(AD-BC)^2} = \langle R(e_1,e_2)e_2,e_1\rangle_g,$$

as desired. □

A special case of Lemma 4.2.2 occurs when (M,g) is a Riemannian manifold of dimension 2. Then if $p \in M$, we have that T_pM is itself a two dimensional subspace of T_pM. Taking $\mathcal{P} = T_pM$ in Lemma 4.2.2 then yields the following statement.

Corollary 4.2.3. *If (M,g) is a Riemannian manifold of dimension 2, then at each point $K(u,v)$ is independent of u,v.*

Example 4.2.4. *Consider the hyperbolic space $\mathbb{H}^2 = \{(x,y) | y > 0\}$ with its usual metric $g = \begin{pmatrix} y^{-2} & 0 \\ 0 & y^{-2}\end{pmatrix}$, and let p be a point in $\mathbb{H}^2$.*

This Riemannian manifold has dimension 2. Thus, by Corollary 4.2.3, $K(u,v)$ is constant over all linearly independent pairs $u,v \in T_p\mathbb{H}^2$. One possible pair (u,v) is indeed (∂_1,∂_2), so we can say $K(u,v) = K(\partial_1,\partial_2)$ for all $u,v \in T_p\mathbb{H}^2$. But we can now explicitly compute this:

$$K(\partial_1,\partial_2) = \frac{\langle R(\partial_1,\partial_2)\partial_2,\partial_1\rangle_g}{\langle \partial_1,\partial_1\rangle_g\langle \partial_2,\partial_2\rangle_g - \langle \partial_1,\partial_2\rangle_g^2}.$$

From the metric components, we have $\langle \partial_1, \partial_1 \rangle_g = \langle \partial_2, \partial_2 \rangle_g = y^{-2}$ *and* $\langle \partial_1, \partial_2 \rangle_g = 0$*. Also, from Example 3.3.5, we have that* $R(\partial_1, \partial_2)\partial_1 = -y^{-2}\partial_1$*, meaning*

$$\langle R(\partial_1, \partial_2)\partial_2, \partial_1 \rangle_g = \langle -y^{-2}\partial_1, \partial_1 \rangle = -y^{-2}\langle \partial_1, \partial_1 \rangle = -y^{-2}(y^{-2}).$$

Thus,

$$K(\partial_1, \partial_2) = \frac{-y^{-2}(y^{-2})}{(y^{-2})(y^{-2}) - 0^2} = -1.$$

We conclude that at every point p*, the sectional curvature of hyperbolic space is* -1*.*

Example 4.2.4 establishes hyperbolic space $\mathbb{H}^2$ as a simple example of a space with negative sectional curvature. This is an idea we will focus on in the next couple of sections.

4.3. **Jacobi Fields (Milind).** Now that we have discussed geodesics and curvature, we come to an idea that connects these two concepts. That is the notion of a Jacobi field.

Definition 4.3.1. (Jacobi Field). *Let* (M, g) *be a Riemannian manifold with Riemann curvature tensor* R*. Suppose* $\gamma : I \to M$ *be a geodesic for some open interval* I*. A function* J *that sends each* $t \in I$ *to a tangent vector* $J(t) \in T_{\gamma(t)}M$ *is called a Jacobi field on* γ *if the Jacobi equation*

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J(t) + R(J(t), \dot\gamma(t))\dot\gamma(t) = 0$$

holds for all $t \in I$*.*

Jacobi fields clearly have a close relationship with the Riemann curvature tensor R seeing as R appears in the Jacobi equation. Jacobi fields also have a few other interesting properties.

Proposition 4.3.2. *The Jacobi equation is a second-order, linear ordinary differential equation.*

Proof. Recall that ∇ is a first-order linear differential operator, so $\nabla_{\dot\gamma}\nabla_{\dot\gamma}$ is a second-order linear differential operator. Thus, $\nabla_{\dot\gamma}\nabla_{\dot\gamma}J(t)$ is an expression linear in J in terms of its second, first and zeroth derivatives. Similarly, since R is a second-order linear differential operator, $R(J(t), \dot\gamma(t))\dot\gamma(t)$ is linear in J and is a function of its second, first and zeroth derivatives.

Hence, the entire expression $\nabla_{\dot\gamma}\nabla_{\dot\gamma}J(t) + R(J(t), \dot\gamma(t))\dot\gamma(t)$ is linear in J and involves only its second, first and zeroth derivatives. The Jacobi equation is an assertion setting this expression to 0, so it is a second-order linear differential equation. It only involves one independent variable, t, so it is ordinary too. □

One particularly important property of Jacobi fields relates it to geodesics in a fundamental way. This is the next lemma.

Lemma 4.3.3. (Jacobi Fields in a Family of Geodesics). *Let* (M, g) *be a Riemannian manifold with Riemann curvature tensor* R*. Suppose* I_1 *and* I_2 *are open intervals, and to each* $s \in I_2$ *assign a geodesic* $\gamma_s : I_1 \to M$*. Further suppose that this family of geodesics is smooth in* s*. If* $J_s(t) = \partial_s\gamma_s(t)$*, then* J_s *is a Jacobi field on* γ_s *for each* $s \in I_2$*.*

Proof. We will abbreviate $\gamma_s(t)$ as γ where appropriate. From the definition of a geodesic, we have that $\nabla_{\partial_t\gamma}(\partial_t\gamma) = 0$. Now apply $\nabla_{\partial_s\gamma}$ to both sides to obtain

$$\nabla_{\partial_s\gamma}\nabla_{\partial_t\gamma}(\partial_t\gamma) = 0.$$

But from the definition of the Riemann curvature tensor, we have

$$R(\partial_s\gamma, \partial_t\gamma)(\partial_t\gamma) = \nabla_{\partial_s\gamma}\nabla_{\partial_t\gamma}(\partial_t\gamma) - \nabla_{\partial_t\gamma}\nabla_{\partial_s\gamma}(\partial_t\gamma) - \nabla_{[\partial_s\gamma,\partial_t\gamma]}(\partial_t\gamma).$$

The first term of the right-hand side is 0 due to the geodesic equation. We also can compute the Lie bracket in the third term. Observe that for any $f \in C^\infty(M)$

$$[\partial_s\gamma, \partial_t\gamma]f = (\partial_s\gamma)((\partial_t\gamma)f) - (\partial_t\gamma)((\partial_s\gamma)f) = (\partial_s\gamma)(\partial_t f) - (\partial_t\gamma)(\partial_s f) = \partial_s\partial_t f - \partial_t\partial_s f = 0.$$

Thus, $\nabla_{[\partial_s\gamma,\partial_t\gamma]}(\partial_t\gamma) = 0$ as well. We then have

$$R(\partial_s\gamma, \partial_t\gamma)(\partial_t\gamma) = -\nabla_{\partial_t\gamma}\nabla_{\partial_s\gamma}(\partial_t\gamma).$$

This is equivalent to the Jacobi equation with $J = \partial_s\gamma$ if we show that $\nabla_{\partial_s\gamma}(\partial_t\gamma) = \nabla_{\partial_t\gamma}(\partial_s\gamma)$. Thus, it remains to prove this fact.

Using Lemma 3.3.3 we can write the statement as, for each k

$$\begin{aligned} \sum_{i,j}\Gamma^k_{ij}(\partial_s\gamma)_i(\partial_t\gamma)_j + \sum_j(\partial_s\gamma)_j\partial_j(\partial_t\gamma)_k &= \sum_{i,j}\Gamma^k_{ij}(\partial_t\gamma)_i(\partial_s\gamma)_j + \sum_j(\partial_t\gamma)_j\partial_j(\partial_s\gamma)_k \\ \iff \sum_j(\partial_s\gamma)^j\partial_j(\partial_t\gamma)^k &= \sum_j(\partial_t\gamma)^j\partial_j(\partial_s\gamma)^k \\ \iff \left(\sum_j(\partial_s\gamma)_j\partial_j\right)(\partial_t\gamma)_k &= \left(\sum_j(\partial_t\gamma)_j\partial_j\right)(\partial_s\gamma)_k. \end{aligned}$$

But the operator $\sum_j(\partial_s\gamma)_j\partial_j$ is precisely ∂_s, and likewise $\sum_j(\partial_t\gamma)_j\partial_j = \partial_t$. Hence, it suffices to show $\partial_s(\partial_t\gamma)_k = \partial_t(\partial_s\gamma)_k$, which is clearly true as ∂_s and ∂_t commute. □

4.4. **Periodic Jacobi Fields and Negative Curvature (Milind).** One particularly important context to consider Jacobi fields in is the context of periodic geodesics.

Definition 4.4.1. (Periodic Geodesic). *Let M be a smooth manifold and let $\gamma : I \to M$ be a geodesic for some open interval I. Then γ is a periodic geodesic if there is a real $T > 0$ such that $\gamma(t) = \gamma(t+T)$ for all t such that $t, t+T \in I$.*

In essence, a periodic geodesic is one that loops back to where it began. Having defined periodic geodesics, we can define what it means for a Jacobi field to be periodic.

Definition 4.4.2. (Periodic Jacobi Field). *Let (M, g) be a Riemannian manifold and let $\gamma : I \to M$ be a periodic geodesic on M for some open interval I. A Jacobi field J on γ is called a periodic Jacobi field if there is a real $T > 0$ such that $J(t) = J(t+T)$ and $(\nabla_{\dot\gamma}J)(t) = (\nabla_{\dot\gamma}J)(t+T)$ for all t with $t, t+T \in I$.*

In fact, in the special case of a periodic geodesic, we have an important theorem, which asserts that a nonzero periodic Jacobi field cannot exist on a manifold of negative sectional curvature.

Theorem 4.4.3. (Periodic Jacobi Fields given Negative Sectional Curvature). *Let (M, g) be a Riemannian manifold with negative sectional curvature everywhere. If γ is a periodic geodesic on M and J is a periodic Jacobi field along γ, then $J = c\dot{\gamma}$ for some constant scalar c.*

Proof. We will perform the following computation.

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\langle J, J\rangle_g = \nabla_{\dot{\gamma}}\left(2\langle\nabla_{\dot{\gamma}}J, J\rangle_g\right) = 2\langle\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J, J\rangle_g + 2\langle\nabla_{\dot{\gamma}}J, \nabla_{\dot{\gamma}}J\rangle_g.$$

From the Jacobi equation, $\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J = -R(J, \dot{\gamma})\dot{\gamma}$. Also so we can conclude

$$\begin{aligned}\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\langle J, J\rangle_g &= \langle\nabla_{\dot{\gamma}}J, \nabla_{\dot{\gamma}}J\rangle_g - \langle R(J, \dot{\gamma})\dot{\gamma}, J\rangle_g \\ &= \langle\nabla_{\dot{\gamma}}J, \nabla_{\dot{\gamma}}J\rangle_g - K(J, \dot{\gamma})\left(\langle J, J\rangle_g\langle\dot{\gamma}, \dot{\gamma}\rangle_g - \langle J, \dot{\gamma}\rangle_g^2\right).\end{aligned}$$

With K being negative for all inputs and $\langle J, J\rangle_g\langle\dot{\gamma}, \dot{\gamma}\rangle_g - \langle J, \dot{\gamma}\rangle_g^2$ and $\langle\nabla_{\dot{\gamma}}J, \nabla_{\dot{\gamma}}J\rangle_g$ both being nonnegative, we have that $\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\langle J, J\rangle_g$ is always nonnegative. But note that $f := \langle J, J\rangle_g$ is a scalar, so we equivalently have that f'' is nonnegative. Also, since $J(0) = J(T)$ and $(\nabla_{\dot{\gamma}}J)(0) = (\nabla_{\dot{\gamma}}J)(T)$, we have that $f(0) = f(T)$ and since $f'(0) = f'(T)$ since $f' = 2\langle J, \nabla_{\dot{\gamma}}J\rangle_g$.

However, we have

$$0 = f'(T) - f'(0) = \int_0^T f''(t)dt.$$

But since $f'' \geq 0$, this means $f''(t) = 0$ for all $t \in [0, T]$. This means

$$\langle\nabla_{\dot{\gamma}}J, \nabla_{\dot{\gamma}}J\rangle_g - K(J, \dot{\gamma})\left(\langle J, J\rangle_g\langle\dot{\gamma}, \dot{\gamma}\rangle_g - \langle J, \dot{\gamma}\rangle_g^2\right) = 0.$$

Since both terms on the left side are nonnegative as previously discussed, they must both in fact be 0 for equality to hold. Hence,

$$\langle\nabla_{\dot{\gamma}}J, \nabla_{\dot{\gamma}}J\rangle_g = 0,$$
$$\langle J, J\rangle_g\langle\dot{\gamma}, \dot{\gamma}\rangle_g - \langle J, \dot{\gamma}\rangle_g^2 = 0.$$

The first equation implies $\nabla_{\dot{\gamma}}J = 0$. The second equation is the equality case of the inequality $\langle J, J\rangle_g\langle\dot{\gamma}, \dot{\gamma}\rangle_g - \langle J, \dot{\gamma}\rangle_g^2 \geq 0$. This occurs when J and $\dot{\gamma}$ are parallel. Hence, $J(t) = c(t)\dot{\gamma}(t)$ for some smooth function c. We now have

$$0 = \nabla_{\dot{\gamma}}J = \nabla_{\dot{\gamma}}(c\dot{\gamma}) = (\dot{\gamma}c)\dot{\gamma} + c\nabla_{\dot{\gamma}}\dot{\gamma} = (\partial_t c)\dot{\gamma},$$

where we used the fact that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ by Definition 3.4.1 and $\dot{\gamma}c = \partial_t c$ since c is a scalar. We now have that $\partial_t c = 0$, so c is constant, as desired. □

Let us now interpret the result of the above theorem. Suppose $\gamma_s(t)$ is a family of periodic geodesics smooth in s on a Riemannian manifold (M, g) with negative sectional curvature everywhere. From Lemma 4.3.3, $\partial_s\gamma_s$ on γ_s is a Jacobi field, so from Theorem 4.4.3, $\partial_s\gamma_s = c\dot{\gamma}_s$ for some scalar constant c. Intuitively, this means that the difference between two nearby geodesics in the family is proportional to the t derivative of a geodesic. Since the t derivative of a geodesic points along the geodesic, this implies that two nearby geodesics are in fact the same aside from scaling. In summary, on a Riemannian manifold with negative sectional curvature, this suggests that there cannot exist two geodesics in a family that are "too close" to each other.

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Appendix A. Topological Spaces

Definition A.0.1 (Topological Space)**.** *A* topology *on a set M is defined as a collection τ of open subsets of U that satisfy the following properties:*

(1) M and $\emptyset$ belong to τ,
(2) any union of sets in τ belong to τ,
(3) any finite intersection of sets in τ belong to τ,

where each member of τ is an open set.

Definition A.0.2 (Topological Space)**.** *A* topological space *is a pair (M, τ) where M is a set and τ is a topology on M.*

Definition A.0.3 (Open and Closed Sets)**.** *Let (M, τ) be a topological space. A set $U \subseteq M$ is* open *if $U \in \tau$. A set $V \subseteq M$ is* closed *if $M \setminus V$ is open.*

Definition A.0.4 (Metric space)**.** *A* metric space *is a pair (X, d) where X is a set and $d : X \times X \to [0, \infty)$ is a function, called a* metric, *such that for all $x, y, z \in X$:*

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Definition A.0.5 (Cauchy sequence)**.** *Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is called a* Cauchy sequence *if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$d(x_m, x_n) < \varepsilon\,, \forall\, m, n \geq N.$$

Definition A.0.6 (Complete metric space)**.** *A metric space (X, d) is called* complete *if every Cauchy sequence in X converges to a limit that is also an element of X.*

Example A.0.7. *In $X = \mathbb{R}$ with the usual distance $d(x, y) = |x - y|$, the sequence*

$$x_n = 1 + \frac{1}{n}$$

is Cauchy, since $|x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \to 0$ as $m, n \to \infty$. It converges to 1, which still lies in $\mathbb{R}$, so $\mathbb{R}$ is a complete metric space.

Appendix B. Code

This is the code for generating geodesics on the hyperbolic plane, which was used to generate diagram 3.5. We vary the ICs (Initial Conditions) to get different geodesics.

```
clear; close all; clc;
t0 = 0;
T = 5;
f = @(t, s) [ ...
    s(3); ...
    s(4); ...
    (2 / s(2)) * s(3) * s(4); ...
    -(1 / s(2)) * (s(3)^2 - s(4)^2) ...
]

function [value, isterminal, direction] = stopWhenYZero(~, s)
    value      = s(2);
    isterminal = 1;
    direction  = -1;
end

opts = odeset('RelTol',1e-9,'AbsTol',1e-12,'Events',@stopWhenYZero);

ICs = [ ...
    0.0   0.1   cos(pi/2)    sin(pi/2);
    0.2   0.1   cos(pi/3)    sin(pi/3);
    .7   0.1   cos(.9*pi/2)  sin(.9*pi/2)
];

figure('Position',[100 100 600 600]); hold on;
colors = lines(size(ICs,1));

for k = 1:size(ICs,1)
    y_init = ICs(k,:).';
    [ts, ys] = ode45(f, [t0 T], y_init, opts);
    x = ys(:,1); y = ys(:,2);
    plot(x, y, 'LineWidth', 1.5, 'Color', colors(k,:));
end

yline(0, '--k', 'y=0', 'LabelHorizontalAlignment','left');
xlabel('x'); ylabel('y');
```

```
title('Geodesics in the Hyperbolic Plane (MATLAB)');
grid on; axis equal;
legend({'Vertical','Semi 1','Semi 2'}, 'Location','best');
xlim([-1 3]); ylim([0 2]);
```

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