Pick's Formula and Euler's Theorem

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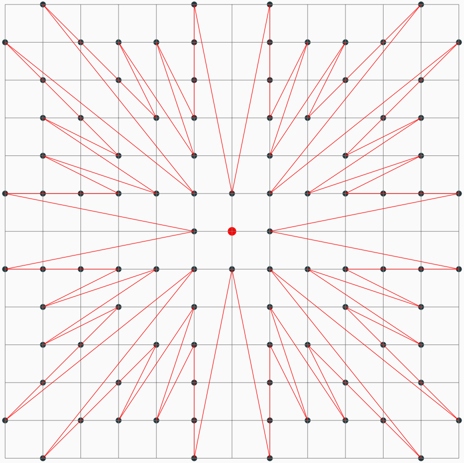
How do you find the area of this?

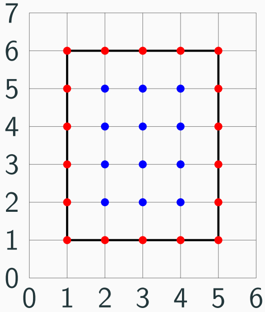


Lattice Point-Area Relation

Lattice Point-Area Relation

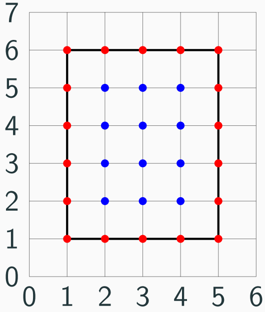
- When first staring at that monstrosity of a star, you might have thought of using the traditional method $L \times W$ for some hidden length L and width W.
- Clearly, trying to approximate an area using similar methods with some shapes can be very difficult.
- With that in mind, lets try to intuitively find a different way to approximate an area; namely, through lattice points!

Lattice Points in a Rectangle



- Take a rectangle R_l with dimensions (x_r, y_r). Within a given rectangle constructed of horizontal and vertical lines, we see the number of lattice points L can be expressed as $L = (x_r + 1) \times (y_r + 1)$.

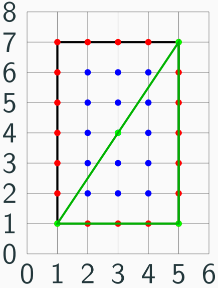
Lattice Points in a Rectangle



- Based on this intuitive observation, we can immediately relate the area of a rectangle r with length and width (x_r, y_r) to the number of lattice points inside of it. In this specific case, we see that

$$Area(r) = L - (2x_r y_r + 1).$$

Lattice Points in a Right Triangle

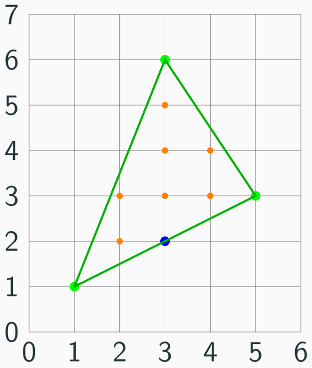


- Can we generalize to a right triangle T_r? If we take the 2 non-corner vertices and label them (x_1, y_1) and (x_2, y_2), we can say that the of lattice points in and the area of T_r respectively, are:

$$T_r = L - (\lfloor \frac{x_r y_r}{2} \rfloor) + GCD(|x_1 - x_2|, |y_1 - y_2|) \quad and$$

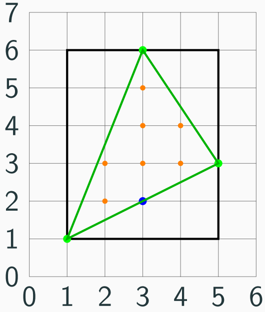
$$A(T_R) = A(R_l) - \frac{A(R_l)}{2}.$$

Triangle on Rectangle Border



Lets see if we can repeat this process for triangles, a simpler shape than a square. We know that it has a finite number of boundary and interior points, but can we use the combined total of lattice points to express this triangles area?

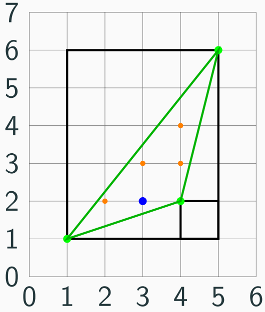
Triangle on Rectangle Border



If we draw the smallest possible imaginary rectangle composed of vertical and horizontal lines R with dimensions (x, y) around this triangle T, and let T_1, T_2, T_3 be the corner triangles made by the two, then we can say

$$A(T) = A(R) - (A(T_1) + A(T_2) + A(T_3)).$$

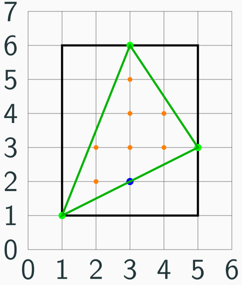
Obtuse Triangle on Rectangle Border



Furthermore, we see in an obtuse scenario that we get a similar composition of known areas, but this time with an included rectangle r_s in our area subtraction, giving equation

$$A(T) = A(R) - (A(T_1) + A(T_2) + A(T_3) + A(r_s)).$$

Triangle on Rectangle Border



From our previous area-lattice point relations, we see that the area of any triangle can be represented by it's interior lattice points. The algorithm representing this relation takes too long to explain, but it obeys the predefined relation expectations. Now, with this differing line of thought, we can consider the generalized lattice point formula involving this relation, namely:

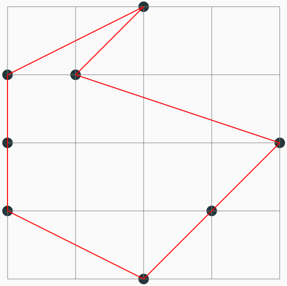
Pick's Formula

Formula (Pick's)

For any polygon with vertices on lattice points, suppose there are I interior points, B boundary points with respect to the polygon, then its area is given by

$$I + \frac{B}{2} - 1.$$

Pick's Formula Example

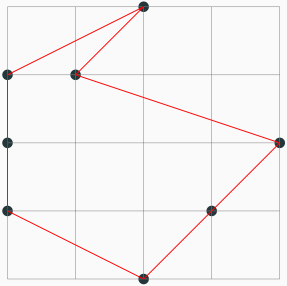


- Here we see that $I = 5$ and $B = 8$, so:

$$Area = I + \frac{B}{2} - 1 = 5 + \frac{8}{2} - 1 = 8.$$

- From this formula, we can see that the area of the Farey sunburst from earlier is $1 + \frac{96}{2} - 1 = 48$.

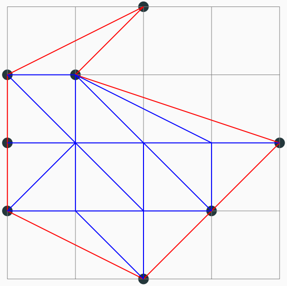
Pick's Formula Example



- Before we show that this generalized proof works for arbitrary polygons, there are a few more concepts we need to introduce.

Triangulation

Triangulation Example



Triangulation

Definition (Primitive Lattice Triangle)

A triangle that has no interior or boundary points other than lattice-point vertices.

Definition (Triangulation)

"Triangulation" is the process of dividing a polygon into a set of pairwise or almost disjoint triangles using line segments (diagonals) that connect lattice points.

Triangulation

We want to prove we can triangulate any polygon. For the sake of contradiction, let $P = A_1A_2\dots A_{k-1}A_k$ be the polygon with the least number of sides that cannot be triangulated with $k > 3$. For this to be a polygon there must exist some angle $\angle A_iA_{i+1}A_{i+2} < 180^\circ$. From here we have 2 cases.

Case 1: there exists a point inside $\triangle A_iA_{i+1}A_{i+2}$.

Case 2: there doesn't exist a point inside $\triangle A_iA_{i+1}A_{i+2}$.

Triangulation

Similarly we want to prove we can triangulate any triangle into primtive lattice point triangles. For the sake of contradiction, let $\triangle XYZ$ be some triangle with interior or boundary points that cannot be triangulated further.

Case 1: Interior point.

Case 2: Boundary point.

Farey Sequence

Farey Sequence

- To gain better relations with our area expression formulas, we use the following property from Farey sequences:
  If $\frac{a}{b}, \frac{c}{d}$ are consecutive fractions in a Farey sequence, then $ad - bc = -1$.
- For example, this is true in F_4:

$$\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}.$$

- Goal: to show each primitive lattice triangle has area $\frac{1}{2}$.

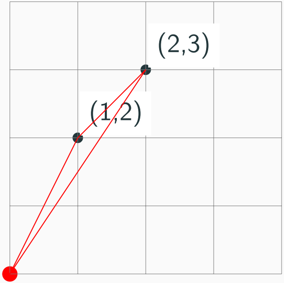
Farey Sequence

Lemma

There are no lattice points in the triangle formed by (a, b), (c, d) *and* $(0, 0)$ *where* $\frac{a}{b}$, $\frac{c}{d}$ *are adjacent terms in a Farey sequence.*

- Suppose FTSOC that there is a point that lies inside this triangle. WLOG let $b < d$. Then the slope of this point would need to lie between $\frac{b}{a}$ and $\frac{d}{c}$ and have denominator at most $\max(a, c)$, which contradicts the claim that $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent terms in a Farey sequence.

Example!!

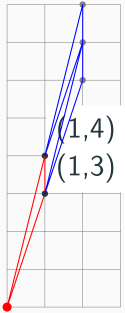


$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$

Slopes: $\frac{3}{2}, 2$

Farey Sequence

By our previous lemma, there are no points in the triangle or on the boundary of the triangle enclosed by $(0,0),(a,b),(c,d)$ up to translation.



$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$

Farey Sequence

Lemma

Adjacent terms in a Farey sequence $\frac{a}{b}, \frac{c}{d}$ satisfy $ad - bc = -1$.

- By our previous lemma, the next possible term that is in between two adjacent terms, $\frac{a}{b}, \frac{c}{d}$, from F_k is $\frac{a+c}{b+d}$.
- Check that
  $a(b+d) - b(a+c) = d(a+c) - c(b+d) = ad - bc = -1.$

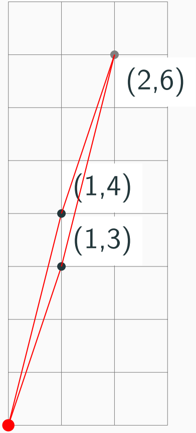
Farey Sequence

Lemma

A primitive lattice triangle has area $\frac{1}{2}$.

- A parallelogram formed by $\vec{u} = \langle a, b \rangle$ and $\vec{v} = \langle c, d \rangle$ has an area of $|ad - bc|$.
- We have shown that the next possible lattice point with slope between $\frac{b}{a}$ and $\frac{d}{c}$ is $(a + c, b + d)$, so there are no interior points in this parallelogram. So, it is made up of two primitive lattice triangles, which completes our proof.

Example!!



$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$

Pick's Formula

Proof of Pick's Formula

Reminder: Pick's formula says that the area of a polygon with I interior lattice points and B boundary lattice points is $I + \frac{B}{2} - 1$.

Proof Outline. To find the area of a polygon, first split its interior into primitive lattice triangles. Since each triangle has area $1/2$, the area of the polygon is $(\frac{1}{2}) \times T$ where T its number of triangles inside the polygon. This relation between areas begs the question: how can we express the number of primitive lattice triangles in a polygon based on a polygon's interior and boundary points?

We claim there are $T = 2I + B - 2$ of them, and we will prove it by induction.

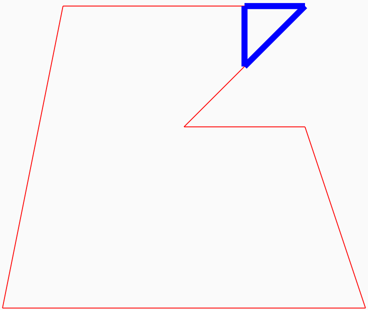
Proof of Pick's formula

Claim: The number of primitive triangles in a polygon is $T = 2I + B - 2$.

Base case. When $T = 1$, the claim is true because $T = 2 \times 0 + 3 - 2 = 1$.

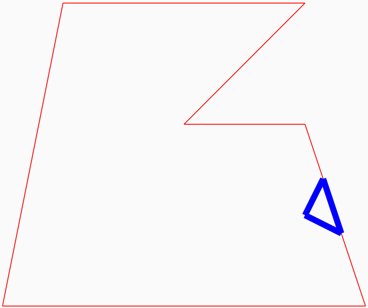
Inductive step. Assume that the claim is true for any polygon with $T = n$. Now take any polygon P with $T = n + 1$. Remove one primitive lattice triangle from it to create a polygon P' made of $T' = n$ primitive lattice triangles.

Proof of Pick's formula



If we removed a triangle with 3 vertices on the boundary of P, then $I' = I$, $B' = B - 1$, and the claim holds for P since $(T' + 1) = 2(I') + (B' + 1) - 2$.

Proof of Pick's formula



If we removed a triangle with 2 vertices on the boundary of P, then $I' = I - 1$, $B' = B + 1$, and the claim holds for P since $(T' + 1) = 2(I' + 1) + (B' - 1) - 2$.

Euler's Theorem

Euler's Theorem

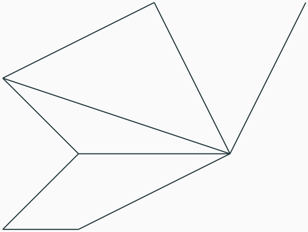
Theorem (Euler's)

In a nonempty connected graph in plane, suppose there are V vertices, E edges and F faces. Then

$$V - E + F = 2.$$

This is saying that 2 is the "Euler characteristic" of the plane. Here the surfaces that we're finding the Euler characteristic of are areas of the plane (not shapes in 3d like a torus, although you can talk about the Euler characteristic of that too).

Example of Euler's theorem



$$F = 4, \quad E = 9, \quad V = 7$$

$$4 = 9 - 7 + 2$$

Proof of Euler's theorem

We did this by induction on the number of edges.

- **Base case.** The graph is just one vertex and no edges. The claim holds because there is one face, the "outside face", and $1 = 0 - 1 + 2$.
- **Inductive step.** Assume that the claim holds for graphs with n edges. Let G be a graph with $E = n + 1$ edges. Then remove one edge from the graph to get a new graph G' with n edges. Let F', E', V' be the numbers of faces, edges and vertices of G', respectively. By the assumption, $F' = E' - V' + 2$.

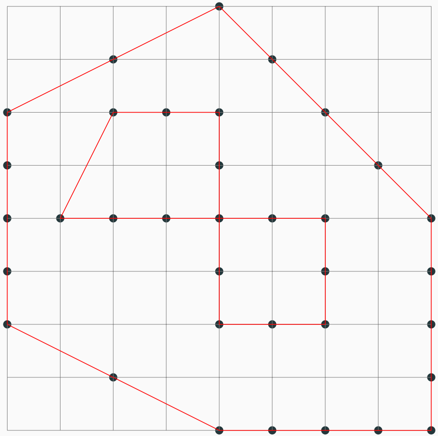
Proof of Euler's Theorem

There are two types of edges that we could remove.

- **Type 1.** A "pendant" edge (that has one end where it is not connected to any other edges). Removing such an edge doesn't change the number of faces, and removes one vertex and one edge. So $F' = F$, $V' = V - 1$, and $E' = E - 1$. So
$$F = E - V + 2.$$
- **Type 2.** A "cycle edge" (that is part of a cycle). Removing such an edge merges two faces into one and doesn't change the number of vertices. So $F' = F - 1$, $V' = V$, and $E' = E - 1$. Again,
$$F = E - V + 2$$
and the claim holds for G. $\square$

Generalizing Pick's Formula

Example



$$A(P) = 39, B = 35, I = 21, e' = 36, h = 2, \chi = 2$$

Generalizing Pick's Formula

Define h as the number of holes inside our polygon P. Define e' as the number of edges on the boundary of P. Let T be the number of primitive lattice triangles. We have

$$F = T + h + 1$$

and

$$V = I + B.$$

Additionally from counting the edges in two ways we have

$$3T = 2E - e'.$$

Lastly from Euler's theorem we have

$$F + V - E = \chi.$$

Combining all of these together we have

$$A(P) = I + B - \frac{e'}{2} + h + 1 - \chi.$$

Polygon Intersection

Polygon Intersection

- From the shown properties of triangulation of polygons and Pick's theorem with simple, single-face polygons, we can make a broader extension to the intersection of 2 polygons.
- Let P and Q be lattice point polygons with intersection X and union Y.

  Through fluid manipulation of areas via Cavalieri's Principle for 2 dimensions, we can deduce that

  $$A(P) + A(Q) = A(X) + A(Y)$$

  where $A(P)$ and $A(B)$ is the area function of some arbitrary polygon B.

Polygon Intersection

- Similarly, where I_A and B_A are the number of interior and boundary points for an lattice polygon A, we can deduce that

$$I_P + I_Q = I_X + I_Y$$

and that

$$B_P + B_Q = B_X + B_Y.$$

- These clean intersection and union formulas stem from our original relations involving polygon area, boundary points, and interior points, and, with our other relations, allow us to generalize to a combination of many polygons.

Conclusion

To conclude, by using the fundamental properties, relations, formulas and theorems discussed today, we can connect seemingly unrelated properties of shapes, which can expand the ways we can manipulate polygons—or algebraic structures adjacent to polygons—within $2D$ planes.

Thank you for your time!

Any questions?