PICK'S FORMULA AND EULER'S THEOREM

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Abstract. Pick's Formula is an efficient way to compute the area of a lattice polygon and it has numerous connections to number theoretical subjects. In this paper, we discuss how these properties all relate among each other through our manipulation of lattice-point polygons, largely in the Cartesian plane, but with brief touches on 3D space as well. The contents will dive into Pick's Formula and Euler's Theorem, a useful property relating components of a polygon. We are interested in inductive proofs of Pick's Formula and Euler's Theorem, where we will define ways to fundamentally break down any kind of lattice point polygon into the smallest possible components through triangulation, as well as how these smaller components relate to fundamental factors such as slope and the Farey sequence. Finally, after proving Pick's Formula and Euler's Theorem, we explore ways that they overlap, as well as how they work together when multiple polygons are introduced into a space.

1. Introduction

Shapes in the 2D or 3D spaces have a variety of components that give them measurable quantities. Edges, vertices, area, orientation, all of these can give us different facets of information regarding a shape's properties, as well as the properties of other structures that can be organized into said shapes.

However, while some of these components have some well-known relations, others have less obvious relations. For example, is there an immediately apparent way to relate a rectangle's length and height to its area? Intuitively, we know that the hight times the width gives the area, but what about other shape components? Is there a way to relate a shape's perimeter to the number of interior points it contains? What about the number of lattice points contained in and on a 3D shape that a given plane intersects to the properties of the faces of said shape?

As we find new components and properties of different shapes, the train of thought between their connections can appear blurred, or even non-existent. Some properties of shapes have yet to be connected today.

This paper was made with the goal of creating more non-intuitive, proof-based connections between shape components and properties, and seeing what new pathways for thought they reveal about different structures with a large emphasis on our primary focus throughout the paper: lattice polygons.

2. Definitions

Definition 2.1 (Simple polygon)**.** *A simple polygon is a closed shape formed by a connected boundary that consists of a finite number of line segments and does not intersect itself.*

Definition 2.2 (Lattice polygon)**.** *A lattice polygon is a polygon with all vertices on lattice points of the plane.*

Definition 2.3 (Interior point)**.** *An interior point is a lattice point that is bounded strictly inside a polygon.*

Definition 2.4 (Boundary point)**.** *A boundary point is a lattice point that lies on an edge of a simple polygon.*

Definition 2.5 (Almost disjoint)**.** *We say two triangles are almost disjoint if they are either disjoint or intersect only along a common edge or vertex.*

Definition 2.6 (Primitive lattice triangle)**.** *A primitive lattice triangle is a lattice triangle that has no interior points or boundary points other than vertices.*

Definition 2.7 (Triangulation)**.** *Call the process of dividing a polygon into a set of pairwise or almost disjoint triangles using line segments (diagonals) that connect lattice points triangulation.*

Remark 2.8. *We use $A(P)$ to denote the area of a polygon P.*

3. Farey sequences and primitive lattice triangles

Lemma 3.1. *There are no lattice points in the triangle formed by (a, b), (c, d) and $(0, 0)$ where $\frac{a}{b}$, $\frac{c}{d}$ are adjacent terms in a Farey sequence.*

Proof. Suppose for the sake of contradiction that there is a point that lies inside this triangle. Without loss of generality assume $b < d$. Let this point has coordinates, (p, q) where $b < q < d$.

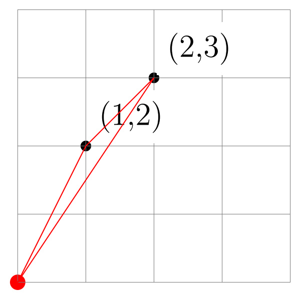

FIGURE 1. An example of a primitive lattice triangle. Notice that the slopes of the two segments from the origin are 2 and $\frac{3}{2}$. The reciprocals are adjacent terms in the Farey sequence $F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$.

Then, the slope of this point is $\frac{q}{p}$, which lies between the slopes $\frac{b}{a}$ and $\frac{d}{c}$, giving us

$$\frac{d}{c} < \frac{q}{p} < \frac{b}{a} \implies \frac{a}{b} < \frac{p}{q} < \frac{c}{d}.$$

Since $q < d$, this term should be in our Farey sequence, but that contradicts the assumption that $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent terms in our Farey sequence. So, there is no such (p, q) in this triangle. □

See Figure 1 for an example.

Lemma 3.2. *For adjacent terms in a Farey sequence $\frac{a}{b}$, $\frac{c}{d}$, $\frac{a+c}{b+d}$ is the term with the least denominator satisfying $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.*

Proof. By 3.1, there are no points in the triangle or on the boundary of the triangle enclosed by $(0, 0), (a, b), (c, d)$ up to translation.

Consider the triangle composed of four identical triangles as the one with $(0, 0), (a, b), (c, d)$. Then, the only lattice point with slope strictly between $\frac{b}{a}, \frac{d}{c}$ is the point $(a + c, b + d)$ with slope $\frac{b+d}{a+c}$; see Figure 2 for an example illustration. Taking the reciprocal of the slopes yields our result.

□

Lemma 3.3. *Adjacent terms in a Farey sequence $\frac{a}{b}$, $\frac{c}{d}$ satisfy $ad - bc = -1$.*

Proof. Base case. $\frac{0}{1}, \frac{1}{1}$ satisfy $0 \cdot 1 - 1 \cdot 1 = -1$.

Inductive step. For some $k \in \mathbb{N}$, the Farey sequence F_k has all its adjacent term $\frac{a}{b}, \frac{c}{d}$ satisfying $ad - bc = -1$. Consider F_{k+1}.

We know that the next possible term that is in between two adjacent terms, $\frac{a}{b}, \frac{c}{d}$, from F_k is $\frac{a+c}{b+d}$ by 3.2. We do not need to worry about

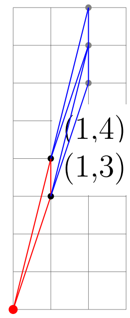

FIGURE 2. Illustration showing that $\frac{a+c}{b+d}$ is the term with the least denominator satisfying $\frac{a}{b} < \frac{a+b}{c+d} < \frac{c}{d}$ for $(a, b, c, d) = (1, 4, 1, 3)$. $\frac{a}{b} = \frac{1}{4}$ and $\frac{c}{d} = \frac{1}{3}$ are adjacent terms in the Farey sequence F_4.

the terms that do not have a new element in between them since their cross-difference remains -1.

So in F_{k+1}, the cross-difference for $\frac{a}{b}, \frac{a+c}{b+d}$ is $ab + ad - ab - bc = ad - bc = -1$. Similarly, $\frac{a+c}{b+d}, \frac{c}{d}$ has cross-difference $ad + cd - bc - cd = ad - bc = -1$.

Since F_{k+1} also satisfies this property then this is true for all $F_n, n \in \mathbb{N}$.

□

Lemma 3.4. *A primitive lattice triangle has area $\frac{1}{2}$.*

Proof. Consider the matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

which has determinant 1 where $\frac{a}{b}, \frac{c}{d}$ are adjacent terms in a Farey sequence.

Applying this matrix as a transformation to the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ preserves the area of the shape, so the parallelogram with vertices $(0, 0)$, (a, b), (c, d) and $(a + c, b + d)$ has area 1; see Figure 3 for an example. There are no interior points in this parallelogram by the previous lemma.

So, the triangle formed by $(0, 0)$, (a, b) and (c, d) has no interior points in it and has half the area of the parallelogram, so the area of the triangle is $\frac{1}{2}$. □

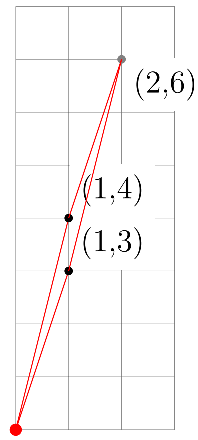

FIGURE 3. Parallelogram formed by applying the transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where $(a, b, c, d) = (1, 4, 1, 3)$, to the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. $\frac{a}{b} = \frac{1}{4}$ and $\frac{c}{d} = \frac{1}{3}$ are adjacent terms in the Farey sequence F_4.

4. Triangulation

In this section, we will show that any polygon can be split into some number of primitive lattice triangles.

4.1. **Primitive Lattice Triangulation.**

Lemma 4.1. *Any simple lattice polygon can be partitioned into primitive lattice triangles.*

Proof. For the sake of contradiction, let $P = A_1A_2 \dots A_{k-1}A_k$ be the polygon with the smallest number of sides $k > 3$ that cannot be triangulated. For this to be a polygon there must exist some angle $\angle A_iA_{i+1}A_{i+2} < 180°$. From here we have 2 cases.

Case 1: there exists a point X inside $\triangle A_iA_{i+1}A_{i+2}$. This means we can draw the line $A_{i+1}X$, which triangulates P into two polygons with fewer than k sides; see Figure 4 for an example. These polygons can be triangulated by the definition of P.

Case 2: there does not exist a point inside $\triangle A_iA_{i+1}A_{i+2}$. This means that the line $\overline{A_iA_{i+2}}$ lies inside P, triangulating P into $\triangle A_iA_{i+1}A_{i+2}$ and a polygon with $k-1$ sides; see Figure 5 for an example. The polygon with $k-1$ sides can be triangulated, resulting in a contradiction. $\square$

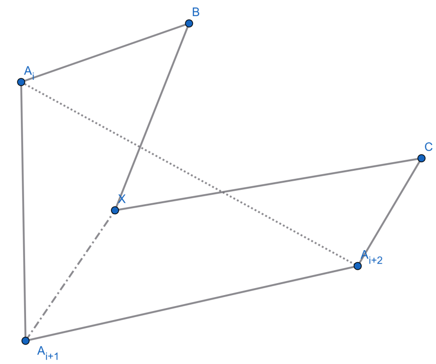

FIGURE 4. Example of a polygon $P = A_1A_2 \ldots A_{k-1}A_k$ with a point X inside $\triangle A_iA_{i+1}A_{i+2}$.

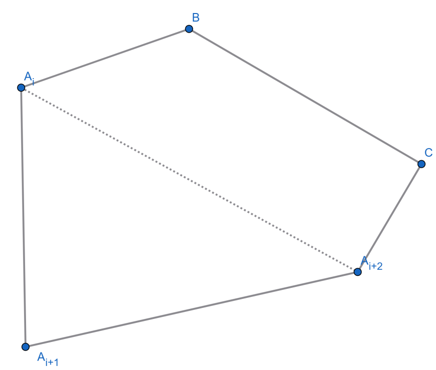

FIGURE 5. Example of a polygon P with no points inside $\triangle A_iA_{i+1}A_{i+2}$.

Lemma 4.2. *Any lattice triangle can be partitioned into primitive lattice triangles.*

Proof. For the sake of contradiction, let $\triangle XYZ$ be some triangle with interior or boundary points that cannot be triangulated further.

Case 1: $\triangle XYZ$ has an interior point I.

Draw the lines $\overline{XI}$, $\overline{YI}$, $\overline{ZI}$, which triangulates $\triangle XYZ$ into $\triangle XYI$, $\triangle YZI$, $\triangle YXI$; see Figure 6 for an example.

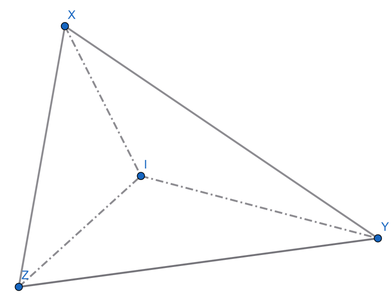

FIGURE 6. Example of a triangle $\triangle XYZ$ with an interior point I.

Case 2: $\triangle XYZ$ has an boundary point B.

Without loss of generality, assume B lies on YZ; see Figure 7 for an example. Draw the line $\overline{XB}$, which triangulates $\triangle XYZ$ into $\triangle XYB$ and $\triangle XZB$, resulting in a contradiction.

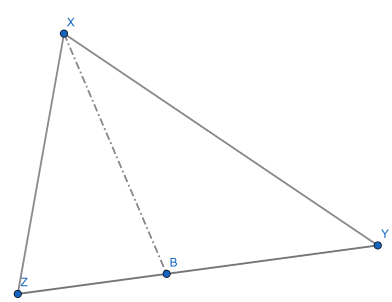

FIGURE 7. Example of a triangle $\triangle XYZ$ with an boundary point B.

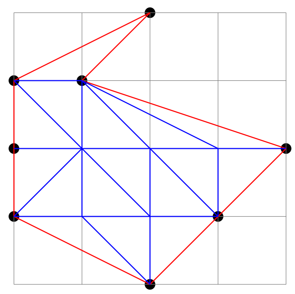

FIGURE 8. Example of a triangulation of a lattice polygon into primitive lattice triangles.

□

This means that any lattice polygon can be split into some number of primitive lattice triangles; see Figure 8 for an example.

4.2. **General Polygon Triangulation.**

Proof. Now, we want to show that we can split any concave polygon into $n-2$ triangles as the maximum bound of triangles it can be split into.

To do this, we want to ensure that we don't add any more vertices in the interior of our polygon to minimize the number of triangles we need to make. Second, we want to have all of our triangles come out of one point and meet every other non-adjacent point. Because for a polygon with n sides, we will make $n-3$ diagonals, which will give us $n-2$ triangles.

Furthermore, because we can find the number of lattice points in any triangle (see General Lattice Point-Area Relations), we can imply that we can represent any concave polygon's area by its lattice points as well!

5. EULER'S THEOREM

Theorem 5.1 (Euler's Theorem)**.** *In a nonempty connected graph in the plane, suppose there are V vertices, E edges and F faces. Then the following relation is satisfied:*

$$V - E + F = 2.$$

Example 5.2. *Consider the graph shown in Figure 9. In the graph, $F = 4$, $E = 9$, and $V = 7$. Theorem 5.1 states that $4 - 9 + 7 = 2$.*

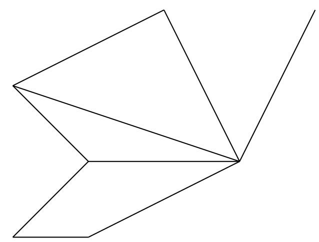

FIGURE 9. Example of a graph with $F = 4$ faces, $E = 9$ edges, and $V = 7$ vertices.

Definition 5.3. *A "tip vertex" in a graph is a vertex with only one edge connected to it.*

Definition 5.4. *A "pendant edge" in a graph is an edge connected to a tip vertex.*

Definition 5.5. *A "cycle edge" in a graph is an edge that is part of a cycle.*

Lemma 5.6. *In any finite graph with at least one edge, there exists at least one cycle edge or pendant edge.*

Proof. Let G be a finite graph with at least one edge. Start at any vertex v and go along any edge connected to that vertex. From there, continue along any edge that we have not yet used. Continue doing this until we can't move along any edge. Since there are finitely many edges, we will not be able to go along new edges forever, so we will eventually have to stop.

If the vertex we end at has any edges apart from the edge we came in along, then (since those edges must already have been used, otherwise we wouldn't have ended there) we have found a cycle and the edge that we come in along is part of that cycle. So there is at least one cycle edge.

If the vertex we end at is a tip vertex, then the edge we came in along is a pendant edge (since it is connected to a tip vertex), and so there is at least one pendant edge.

$\square$

Proof. (Euler's Theorem) Let E, V, and F be the numbers of edges, vertices, and faces of a graph G, respectively.

We will prove this by induction on the number of edges E.

Base case. Consider a graph with $E = 0$. Since there are no edges, there cannot be more than one vertex because the graph must be connected. The graph must have at least one vertex because it is nonempty. So it must have exactly one vertex, so $V = 1$. It also has only one face (the "outside face"), $F = 1$. The claim holds in this case because $1 = 0 - 1 + 2$.

Inductive step. Assume that for some n, the claim holds for every graph with $E = n$. Let G be any graph with $E = n + 1$. Then by lemma 4.4, G has at least one cycle edge or pendant edge. Remove that edge from G to make a new nonempty connected graph G' with $E' = E - 1 = n$ edges. There are two cases.

Case 1: The edge that we remove is a cycle edge. Then removing it breaks a boundary between two faces, merging them into one face so that $F' = F - 1$. The number of vertices stays the same, so $V' = V$. In this case the claim holds for G, because

$$F' = E' - V' + 2$$

$$(F - 1) = (E - 1) - V + 2$$

$$F = E - V + 2$$

Case 2: The edge that we remove is a pendant edge. Then removing the edge removes the tip vertex connected to that edge, so $V' = V - 1$. The number of faces does not change, so $F' = F$. In this case the claim holds for G, because

$$F' = E' - V' + 2$$

$$F = (E - 1) - (V - 1) + 2$$

$$F = E - V + 2$$

□

6. Pick's Formula

Theorem 6.1 (Pick's Formula)**.** *Any simple lattice polygon with I interior points and B boundary points has area given by*

$$I + \frac{B}{2} - 1.$$

Example 6.2. *(Pick's Formula) Consider the lattice polygon P shown in Figure 10. P has $I = 5$ interior points and $B = 8$ boundary points, so*

$$A(P) = I + \frac{B}{2} - 1 = 5 + \frac{8}{2} - 1 = 8.$$

We can also find the area of more complicated polygons. For example, the area of the Farey sunburst of order 6 shown in Figure 6 is $1 + \frac{96}{2} - 1 = 48$.

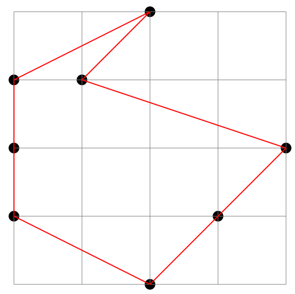

FIGURE 10. Example of a lattice polygon with $I = 5$ interior points and $B = 8$ boundary points.

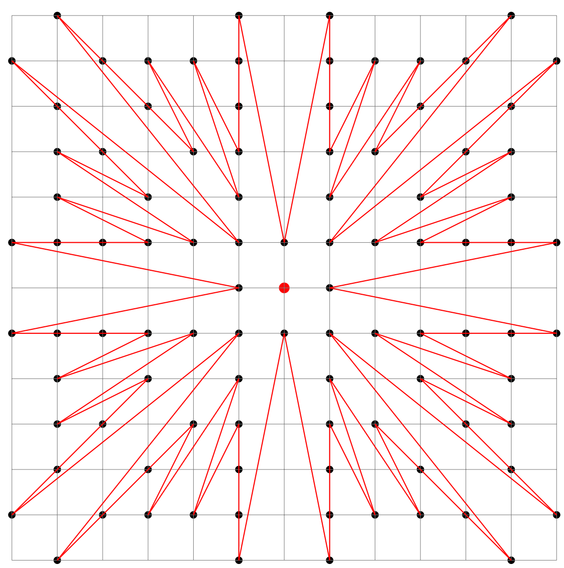

FIGURE 11. Farey sunburst of order 6.

6.1. **An inductive proof of Pick's Formula.** To find the area of a polygon, first split it into primitive lattice triangles. This is possible by lemma 2.7. For a polygon P, let T be the number of primitive lattice triangles in any triangulation of P, I be the number of interior points, B be the number of boundary points, and A be the area of P. By lemma 2.6, each of the triangles has area $\frac{1}{2}$. The primitive lattice triangles in a triangulation do not overlap, so

$$A = \frac{1}{2}T.$$

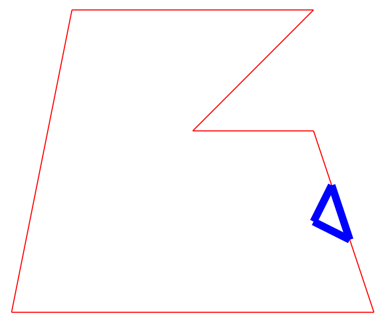

FIGURE 12. Example of a polygon P with a triangle removed. The triangle has two vertices on the boundary of P and one vertex in the interior of P.

Now we need to show that $T = 2I + B - 1$. We will do this by induction on T.

Base case. For $T = 1$, the claim is true because by the definition of a primitive lattice triangle, $I = 0$ and $B = 3$, so $2I + B - 2 = 1 = T$.

Inductive step. Assume that for some n, the claim holds for every polygon with $T = n$. Then take any polygon P with $T = n + 1$. Triangulate it and then remove one primitive lattice triangle from the triangulation in a way that leaves a simple polygon P'. Then P' is a simple polygon with $T' = T - 1$, so by our assumption,

$$T' = 2I' + B' - 2.$$

Now we split into two cases:

Case 1: The triangle that we removed from P had two vertices on the boundary of P and one vertex in the interior of P. An example is shown in Figure 12. Then removing the triangle creates one boundary point and destroys one interior point. So $B' = B + 1$ and $I' = I - 1$ and $T' = T - 1$.

$$\begin{aligned} T' &= 2I' + B' - 2 \\ T - 1 &= 2(I - 1) + (B + 1) \\ T &= 2I + B - 2, \end{aligned}$$

so the claim holds for P.

Case 2: The triangle that we removed from P had three vertices on the boundary of P and no vertices in the interior of P. An example is shown in Figure 13. Notice that removing the triangle creates destroys one boundary point and doesn't change the number of interior points.

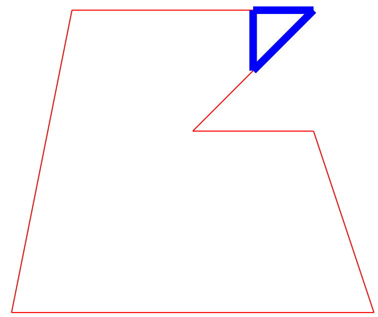

FIGURE 13. Example of a polygon P with a triangle removed. The triangle has three vertices on the boundary of P and no vertices in the interior of P.

So $B' = B-1$ and $I' = I$ and $T' = T-1$. By the inductive assumption,

$$T' = 2I' + B' - 2$$
$$T - 1 = 2(I) + (B - 1)$$
$$T = 2I + B - 2,$$

so the claim holds for P.

These are the only cases that we need to look at, because if a primitive lattice triangle in the triangulation of P has less than two vertices then removing it would leave a polygon that has a hole and is therefore not simple. □

7. General Lattice Point-Area Relation

We could also find a strategy to generalize the area of a triangle from lattice points and, through triangulation, do so for any other polygon by subtracting areas from a rectangle containing said triangle within a coordinate plane.

7.1. **Rectangle Lattice-Area Relation.** Take a rectangle R with dimensions (x_R, y_R), as shown in Figure 14. Within a given rectangle constructed of horizontal and vertical lines, we see the number of lattice points L can be expressed as

$$L = (x_R + 1) \times (y_R + 1).$$

.

Furthermore, we can relate the area of a rectangle R with length and width (x_R, y_R) to the number of lattice points inside of it. In this specific case, we see that

$$A(R) = L - (2x_R y_R + 1),$$

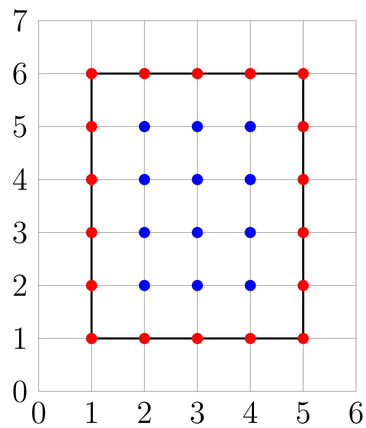

FIGURE 14. Rectangle R with dimensions (x_R, y_R).

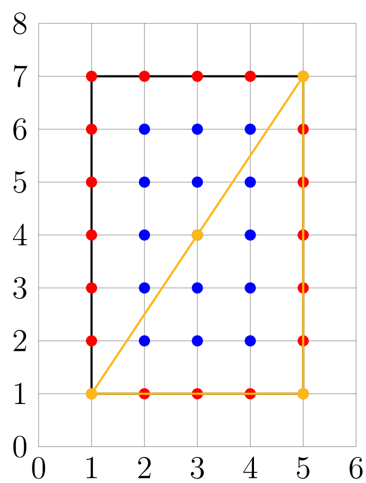

FIGURE 15. Example of a right triangle T (shown in orange) inside a rectangle R (shown in red).

where $A(R) = x_R \cdot y_R$, by the typical definition of the area.

7.2. **Right Triangle Lattice-Area Relation.** To generalize to a right triangle T such as the one shown in Figure 15, we take the vertices < 90 and label them (x_1, y_1) and (x_2, y_2), and take the 90 vertex and label it (x_3, y_3).

. Then, to create a rectangle, whose lattice-area relation is already known, we create a new temporary lattice points at

$$(x_s, y_s),\ (x_s, y_l),\ (x_l, y_s),\ (x_l, y_l)$$

where $x_s = \min(x_1, x_2, x_3),\ x_l = \max(x_1, x_2, x_3),\ , y_s = \min(y_1, y_2, y_3),\ y_l = \max(y_1, y_2, y_3)$.

Create horizontal and vertical lines between each of the points. Another right triangle should be created by the new rectangle, of whose area and lattice points we can subtract from the rectangle to get the area and lattice points in our right triangle.

We can then determine from our rectangle relation that the number of lattice points in and the area of T respectively, are

$$T = L - \left(\left\lfloor \frac{(x_R+1)(y_R+1)}{2} \right\rfloor \right) + \gcd(|x_1 - x_2|, |y_1 - y_2|),$$
$$A(T) = A(R) - \frac{A(R)}{2} = \frac{A(R)}{2},$$

where $x_R = (x_l - x_s)$, $y_R = (y_l - y_s)$, and $A(R) = x_R y_R$.

Although it may seem redundant to write these equations in relation to the original rectangle's lattice points, this will help us with the next two relations.

7.3. **Arbitrary Triangle Lattice-Area Relation.** Because we are able to do area-lattice point relation via subtraction of rectangles and right triangles, we can show that the area of an arbitrary lattice triangle can be determined from known area-lattice point relations.

Consider the acute triangle shown in Figure 16. If we draw the smallest possible imaginary rectangle composed of vertical and horizontal lines R with dimensions (x, y) around this triangle T (as shown in Figure 16), and let T_1, T_2, T_3 be the corner triangles made by the two, then we can say

$$A(T) = A(R) - (A(T_1) + A(T_2) + A(T_3)).$$

Furthermore, we see in an obtuse scenario (see Figure 17 that we get a similar composition of known areas, but this time with an included rectangle r_s in our area subtraction, giving equation

$$A(T) = A(R) - (A(T_1) + A(T_2) + A(T_3) + A(r_s)).$$

Finally, in the scenario in Figure **??**, we can compose the green triangle from the subtraction of two right triangles from a larger rectangle, with the area relation being represented like so:

$$A(T) = A(R) - (A(T_1) + A(T_2)).$$

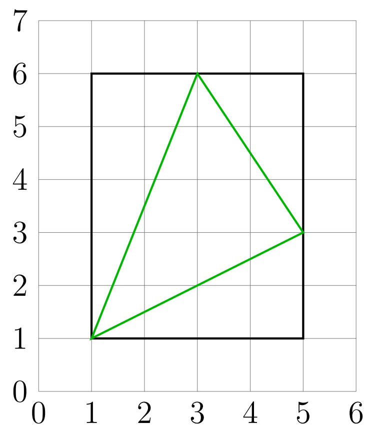

FIGURE 16. Example of an acute lattice triangle.

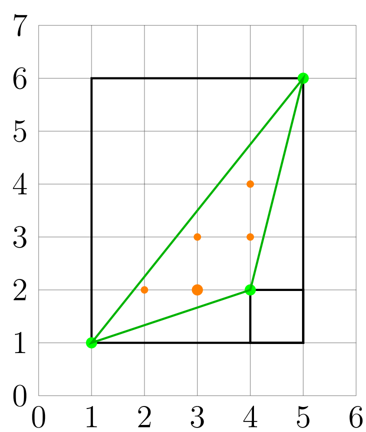

FIGURE 17. Example of an obtuse lattice triangle.

Because we have shown both that a lattice-area relation exists for any triangle composition scenario (situations subtracting 1, 2, or 3 right triangles and/or a rectangle from the larger rectangle) and that, via triangulation, an arbitrary polygon can be split up into a finite number of triangles, we have shown that there does exist such a relation for any kind of polygon.

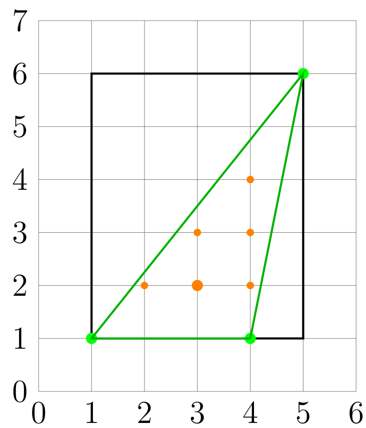

FIGURE 18. Example of an obtuse lattice triangle with a lattice line segment as one side.

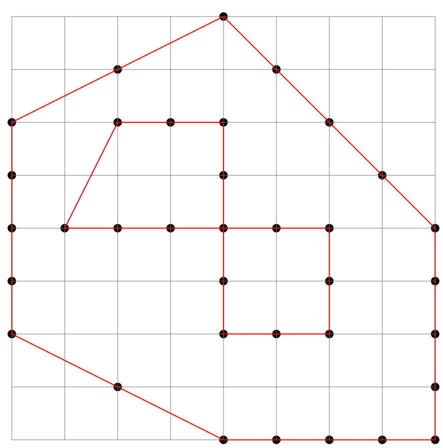

FIGURE 19. Example of a polygon P with $A(P) = 39, B = 35, I = 21, e' = 36, h = 2, \chi = 2$.

8. Generalization of Pick's Formula Using Euler's Theorem

In this section, we consider polygons that may contain holes, such as the one shown in Figure 19. Define h as the number of holes inside our polygon P. Define e' as the number of edges on the boundary of P. Let T be the number of primitive lattice triangles. We have

$$F = T + h + 1$$

and

$$V = I + B.$$

Additionally each primitive lattice triangle has 3 edges that are double-counted except for the ones on the boundary, so we have

$$3T = 2E - e'.$$

Lastly, from Euler's theorem, we have

$$F + V - E = \chi.$$

Combining all of these together, we have

$$A(P) = I + B - \frac{e'}{2} + h + 1 - \chi.$$

This relation can be seen in Figure 19.

9. Translation and Dilation of Simple Lattice Polygons by Integers

Lemma 9.1. *The line between* (a, b) *and* (c, d) *passes through* $\gcd(c - a, d - b)$ *lattice points not including* (a, b).

Proof: The line has slope

$$\frac{c-a}{d-b} = \frac{\frac{c-a}{\gcd(c-a,d-b)}}{\frac{d-b}{\gcd(c-a,d-b)}},$$

meaning we have $\frac{c-a}{\frac{c-a}{\gcd(c-a,d-b)}} = \gcd(c - a, d - b)$ lattice points on the line not including (a, b).

Say we have a lattice polygon P with I_p interior points and B_p boundary points. Let P' be the polygon with the following transformations:

- **Translation:** If we translate a lattice polygon horizontally and vertically by an integer, we can map each lattice point to each other, so $I_p = I_{p'}$ and $B_p = B_{p'}$.
- **Dilation:** Since we can translate without changing I_p and B_p, we can assume that our dilation is centered at $(0, 0)$ with scale factor $k \in \mathbb{Z}$.

Notice that the edge with endpoints (a_1, a_2) and (b_1, b_2) has $\gcd(b_1 - a_1, b_2 - a_2)$ boundary points not including (a_1, a_2). When we apply the dilation by k, the endpoints become (ka_1, ka_2) and (kb_1, kb_2), which has $\gcd(kb_1 - ka_1, kb_2 - ka_2) = k \gcd(b_1 - a_1, b_2 - a_2)$ boundary points not including (ka_1, ka_2). Applying this for all edges, we get

$$kB_p = B_{p'}.$$

With Pick's Formula, we know that

$$A(p) = I_p + \frac{B_p}{2} - 1$$

and

$$k^2 A(P) = A(P') = I_{p'} + \frac{B_{p'}}{2} - 1 = I_{p'} + \frac{kB_p}{2} - 1$$

and solving for $I_{p'}$ we get

$$I_{p'} = (k^2)I_p + (k^2 - k)\frac{B_p}{2} + (1 - k^2).$$

10. Lattice point polygons with intersections and unions

Let P and Q be lattice point polygons in the plane with intersection X and union Y.Through fluid manipulation of areas via Cavalieri's Principle for 2 dimensions, we can deduce that

$$A(P) + A(Q) = A(X) + A(Y).$$

10.1. **Pick's Formula.** Now assume that each connected component of X is a lattice point polygon.

Interior Points: Let I_p be the set of interior points in an arbitrary polygon P. Define I_q, I_x, and I_y similarly. Note $I_x = I_p \cap I_q$ since

$$\forall i \in I_x \iff i \in I_p \text{ and } I_q \iff i \in I_p \cap I_q.$$

Similarly we have $I_y = I_p \cup I_q$ since

$$\forall i \in I_y \iff i \in I_p \text{ or } I_q \iff i \in I_p \cup I_q,$$

which gives us

$$I_y + I_x = I_p + I_q.$$

Boundary Points: Let B_p be the number of boundary points on an arbitrary polygon P. Define B_q, B_x, and B_y similarly.

Note $B_x = B_p \cap B_q$ since

$$\forall i \in B_x \iff i \in B_p, B_q \iff i \in B_p \cap B_q$$
$$\forall i \in B_y \iff i \in B_p \text{ or } B_q \iff i \in B_p \cup B_q,$$

which gives us

$$B_y + B_x = B_p + B_q.$$

10.2. **Euler's Theorem.** As a reminder, Euler's Theorem states that for E edges, V vertices, and F faces, the relation

$$F + V - E = \chi$$

holds for a single polygon, as in this scenario, $\chi = 2$.

If we were to add an arbitrary polygon to this plane with no connection to the original polygon, say, a triangle, then we will have the new relation

$$(F + 1) + (V + 3) - (E + 3) = \chi.$$

When we apply this change with our original polygon's Euler characteristic and simplify, we see that it no longer equals 2, but instead gives us the constant 3.

As shown in the proof for Euler's formula, we see that when we have a single shape, the Euler characteristic remains constant for additions to the shape. However, when we add a new disjoint shape to the plane—and hence add its complete Euler formula to the original one to get an analog of Euler's theorem for the whole plane—the empty plane, which accounts for an extra face to make $\chi = 2$, is counted twice. If we say x is the number of non-connected polygons in a space, and $n = x - 1$, then our previous statement implies that for all x,

$$\chi_x = \chi_n + 1.$$

11. 3D Extension of Pick's Formula

Lemma 11.1. *The volume of a lattice polyhedron is not uniquely determined by the number of interior and boundary lattice points.*

This can be shown if we find two polyhedrons with the same number of interior and boundary points but have different area.

Note that the area of the polyhedron in Figure 20 with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ has 4 boundary points, 0 interior points, and has area

$$\frac{1}{3} \cdot \left(\frac{1}{2} \cdot 1 \cdot 1\right) \cdot 1 = \frac{1}{6}.$$

But also note that the area of the polyhedron in Figure 21 with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 2)$ has 4 boundary points, 0 interior points, and has area

$$\frac{1}{3} \cdot \left(\frac{1}{2} \cdot 1 \cdot 1\right) \cdot 2 = \frac{1}{3}.$$

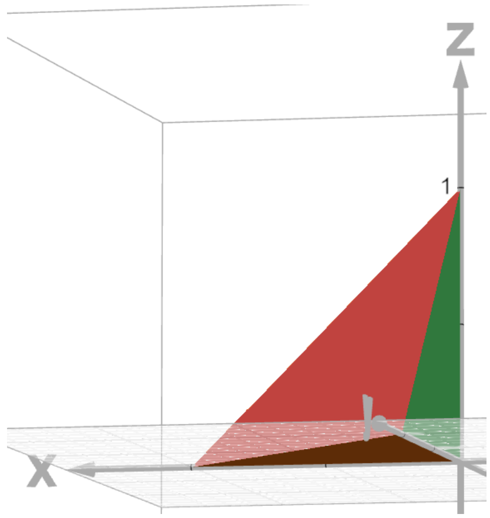

FIGURE 20. Polyhedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

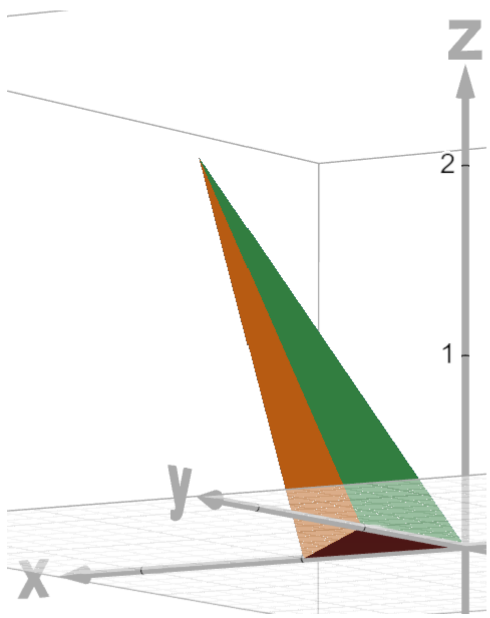

FIGURE 21. Polyhedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 2)$.